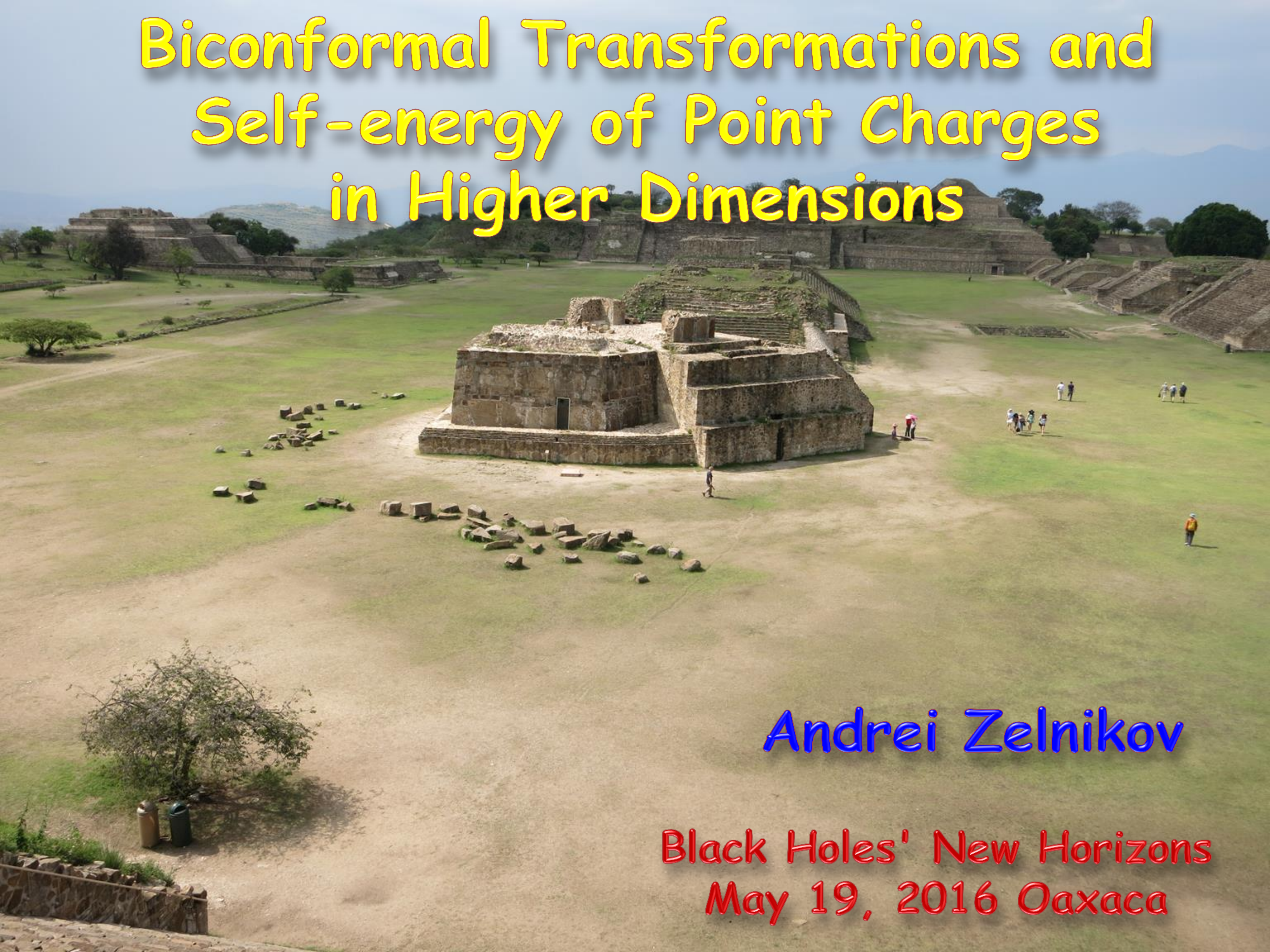
Biconformal Transformations and Self-energy of Point Charges in Higher Dimensions



Andrei Zelnikov

Black Holes' New Horizons
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Based on:

1. Valeri P. Frolov and Andrei Zelnikov, **PHYS REV D91, 064037 (2015)**
2. Valeri P. Frolov and Andrei Zelnikov, **PHYS REV D92, 024023 (2015)**
3. Valeri P. Frolov and Andrei Zelnikov, **JHEP, 1504 (2015)**
4. Valeri P. Frolov, and Andrei Zelnikov: **PHYS REV D 85, 064032 (2012)**
5. Valeri P. Frolov, and Andrei Zelnikov: **PHYS REV D 85, 124042 (2012)**
6. Valeri P. Frolov, and Andrei Zelnikov: **PHYS REV D 86, 044022 (2012)**
7. Valeri P. Frolov, and Andrei Zelnikov: **PHYS REV D 86, 104021 (2012)**

- In quantum electrodynamics the self-energy of an electron diverges and, hence, should be regularized and renormalized. A classical self-energy of a pointlike charge suffers similar divergences. Quantum field theory provides us with methods to deal with this problem systematically.

- We apply QFT methods to fix ambiguities and model dependence of the classical self-energy of charged particles.

- In higher dimensions these problems are much more serious than in four dimensions, and new features appear.

- We consider static scalar charges in the gravitational field of higher dimensional black holes.
- In this case radiation-reaction effects vanish.

In 4 dimensions one can show that a point electric charge gets an additional positive energy due to the self-interaction

[Smith Will (1980); Frolov and Zelnikov (1980,1981); Ritus 1981, Lohiya (1982)].

$$E^{em} = \underbrace{\left(m_{bare} + \frac{e^2}{2\varepsilon} \right)}_{m_{ren}} | g_{00} |^{1/2} + \boxed{\frac{e^2 M}{2\, r^2}}$$

which leads to an additional repulsive (from the black hole) self-force.

In five dimensions self-force of scalar and electric point charges has been calculated by Beach, Poisson, and Nickel (2014)

Unexpected contributions to the self-energy and self-force appear in odd-dimensional spacetimes. This effect is classical but is closely related to quantum conformal anomaly.
For a point scalar charge:

$$\Delta m = -\frac{q^2}{2} G_{\text{reg}}(x,x) = -\frac{q^2}{2} < \varphi^2 >_{\text{ren}}$$

$$\Delta m = -\frac{1}{2} \boxed{\frac{q^2}{\hbar c}} < \varphi^2 >_{\text{(quantum) ren}}$$

$$E = m\sqrt{|g_{00}|}$$

$$< \varphi^2 >_{\text{ren}} \sim 1$$

$$< \varphi^2 >_{\text{(quantum) ren}} \sim \hbar$$

φ is the field defined on (D-1)-dimensional spatial slice (t=const) of a static spacetime

Anomaly of what?

Anomaly of what?

Biconformal symmetry

A scalar massless field Φ in a D-dimensional spacetime

$$\Box \Phi = -4\pi J$$

In the static spaceitime

$$ds^2 = -\alpha^2 dt^2 + g_{ab} dx^a dx^b ,$$

$$X = (t, x^a), \quad \alpha = \alpha(x), \quad g_{ab} = g_{ab}(x).$$

The field equation becomes

$$\hat{F}\,\Phi = -4\pi J\,, \qquad \hat{F} = \frac{1}{\alpha\sqrt{g}} \partial_a \left(\alpha \sqrt{g}\, g^{ab} \partial_b \right).$$

This equation is invariant under the following *bi-conformal* transformations (n=D-3)

$$\Phi = \bar{\Phi}, \quad g_{ab} = \Omega^2 \bar{g}_{ab}, \quad \alpha = \Omega^{-n} \bar{\alpha}, \quad J = \Omega^2 \bar{J},$$

Formally the functional of the self-energy $E = m\sqrt{|g_{00}|}$ **of a charge distribution is invariant under transformations of the static metric.**

$$g_{ab} = \Omega^2(x)\,\bar{g}_{ab}, \qquad g_{00} = \Omega^{-(D-3)}(x)\,\bar{g}_{00}$$

But ***E*** **diverges for point-like sources.**

Regularization breaks this invariance and acquires an anomalous contribution

$$\Delta m = -\frac{q^2}{2} < \varphi^2 >_{\text{ren}}$$

$$\Omega^{(D-3)}\left(< \varphi^2 >_{\text{ren}} + A\right) = \text{const}$$

similar to the conformal anomaly in QFT.

- $$< \varphi^2 >_{\text{ren}} = \Omega^{-n} < \bar{\varphi}^2 >_{\text{ren}} - B$$

$$B(x) = \lim_{x' \to x} \left[G_{\text{div}}(x, x') - \frac{\bar{G}_{\text{div}}(x, x')}{\Omega^{n/2}(x)\Omega^{n/2}(x')} \right].$$

- **In 4D (and any even dimension) the anomaly** ***B=0***
- **In 5D** $B = -\frac{1}{48\pi^2}\Omega^{-3}\Omega^{;c}_{;c} - \frac{\bar{a}_1}{8\pi^2}\Omega^{-2}\ln(\Omega)$

$$A(x) = \frac{1}{288\pi^2} R - \frac{1}{64\pi^2} \ln(g)\, a_1(x)$$

$$a_1(x) = \frac{1}{6} R + V$$

If one starts with a solution of the Einstein equations, a new metric, obtained as a result of these transformations, is not necessarily a solution of the Einstein equations with a physically meaningful stress-energy tensor. However, it may happen that for a specially chosen transformation this new metric has good physical properties and has enhanced symmetries.

An interesting example is a Majumdar-Papapetrou metric, describing the gravitational field of a set of higher dimensional extremely charged black holes in equilibrium. Under a properly chosen biconformal map this metric reduces to the higher-dimensional Minkowski metric. This allows one to solve the static scalar field equation in the Majumdar-Papapertou exactly

$$ds^2 = -U^{-2}\, dt^2 + U^{2/n} \delta_{ab}\, dx^a dx^b \,,$$

$$\Omega = U^{1/n}, \qquad n = D-3,$$

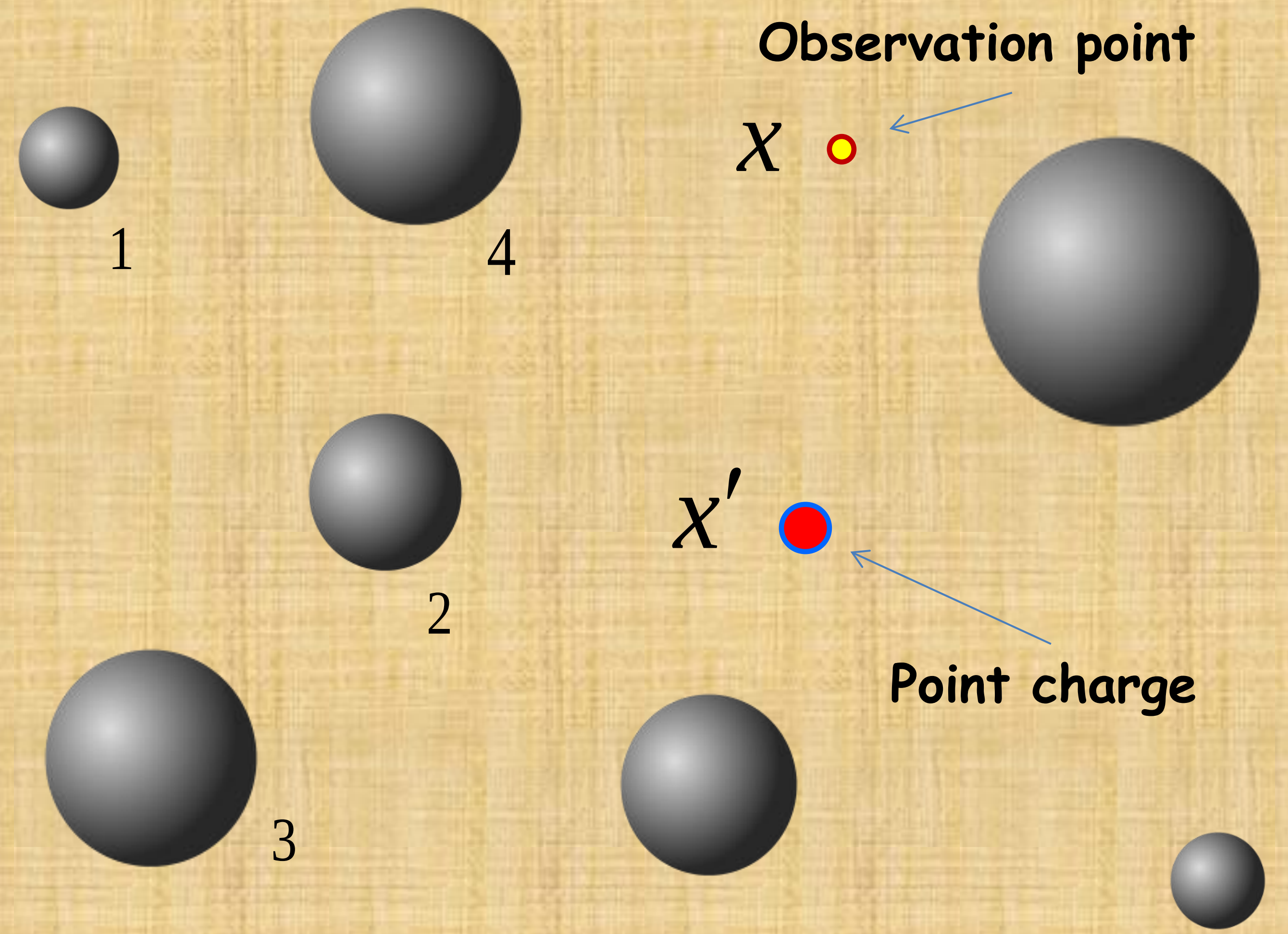
$$d\bar{s}^2 = -dt^2 + \delta_{ab}\, dx^a dx^b \,.$$

$$U = 1 + \sum_k \frac{M_k}{\rho_k^n},$$

$$\rho_k = |\mathbf{x} - \mathbf{x}_k|$$

Higher dimensional Majumdar-Papapetrou metrics

They describe the gravitational field of a set of higher dimensional extremally charged black holes in equilibrium.



In 4D. Near extremal Reissner-Nordström BH

$$E^{em} = \left(m_{bare} + \frac{e^2}{2\varepsilon} \right) | g_{00} |^{1/2} + \frac{e^2 M}{2 r^2}$$

$$E_{self} = e^2 \frac{M}{2r^2}$$

In 5D

$$E_{self} = e^2 \frac{M}{2\pi r^4} + e^2 \frac{M^2 (r^2 - M)}{24\pi r^8}$$

Symmetry enhancement condition

Let us consider an application of the method of the bi-conformal maps to the case of a static spherically symmetric D-dimensional metric.

$$ds^2 = -f(r)\,dt^2 + w^{-1}(r)\,dr^2 + r^2\,d\omega_{n+1}^2\,,$$

$$d\omega_{n+1}^2 = d\theta_n^2 + \sin^2\theta_n\,d\omega_n^2\,,$$

$$d\omega_0^2 = d\phi^2\,.$$

The biconformal transformation with $\Omega = r/a$

$$d\bar{s}^2 = dh^2 + a^2\,d\omega_{n+1}^2\,,$$

$$dh^2 = -\left(\frac{r}{a}\right)^{2n} f(r)\,dt^2 + \frac{a^2}{r^2 w(r)}\,dr^2\,.$$

The scalar curvature of the two-dimensional metric

$$R = -\frac{1}{2a^2 f^2}\left\{ rfw'(2nf + rf') + w[2r^2 f'' - r^2 (f')^2 + 2r(2n+1) ff' + 4n^2 f^2] \right\}.$$

The metric possesses an enhanced symmetry if $R = -\frac{2}{b^2} = const$,
Then the solution reads

$$w = \left(\frac{a^2}{n^2 b^2} + \frac{C}{r^{2n} f} \right) \left(1 + \frac{rf'}{2nf} \right)^{-2}.$$

If the spacetime is asymptotically flat $f = f_0 + f_1 r^{-\gamma} + \ldots$

$\gamma \geq 1.$

Then from the requirement of the absence of a solid angle deficit one gets the condition

$$\frac{a}{nb} = 1.$$

The reference metric: the Bertotti-Robinson spacetime

$$d\bar{s}^2 = a^2\left[\frac{1}{n^2}\left(-(\rho^2-1)\,d\bar{\sigma}^2 + \frac{1}{\rho^2-1}d\rho^2\right) + d\omega_{n+1}^2\right],$$

$$AdS^2 \times S^{(n+1)}$$

$$\rho = \cosh\left(n\int_{r_g}^{r}\frac{dr}{r\sqrt{w(r)}}\right),$$

$$\bar{\sigma} = n r^n a^{-1-n}\,\frac{\sqrt{f}}{\sqrt{\rho^2-1}}\Big|_{r=r_g}\,t\,.$$

Biconformal map of Reissner-Nordström metric to the Bertotti-Robinson spacetime

If $w = f$ **then** $f = 1 - \frac{2M}{r^n} + \frac{Q^2}{r^{2n}}.$

$$r^n = M + \mu\rho, \qquad t = \frac{a^{n+1}}{n\mu}\bar{\sigma}, \qquad \mu = \sqrt{M^2 - Q^2}.$$

R-N

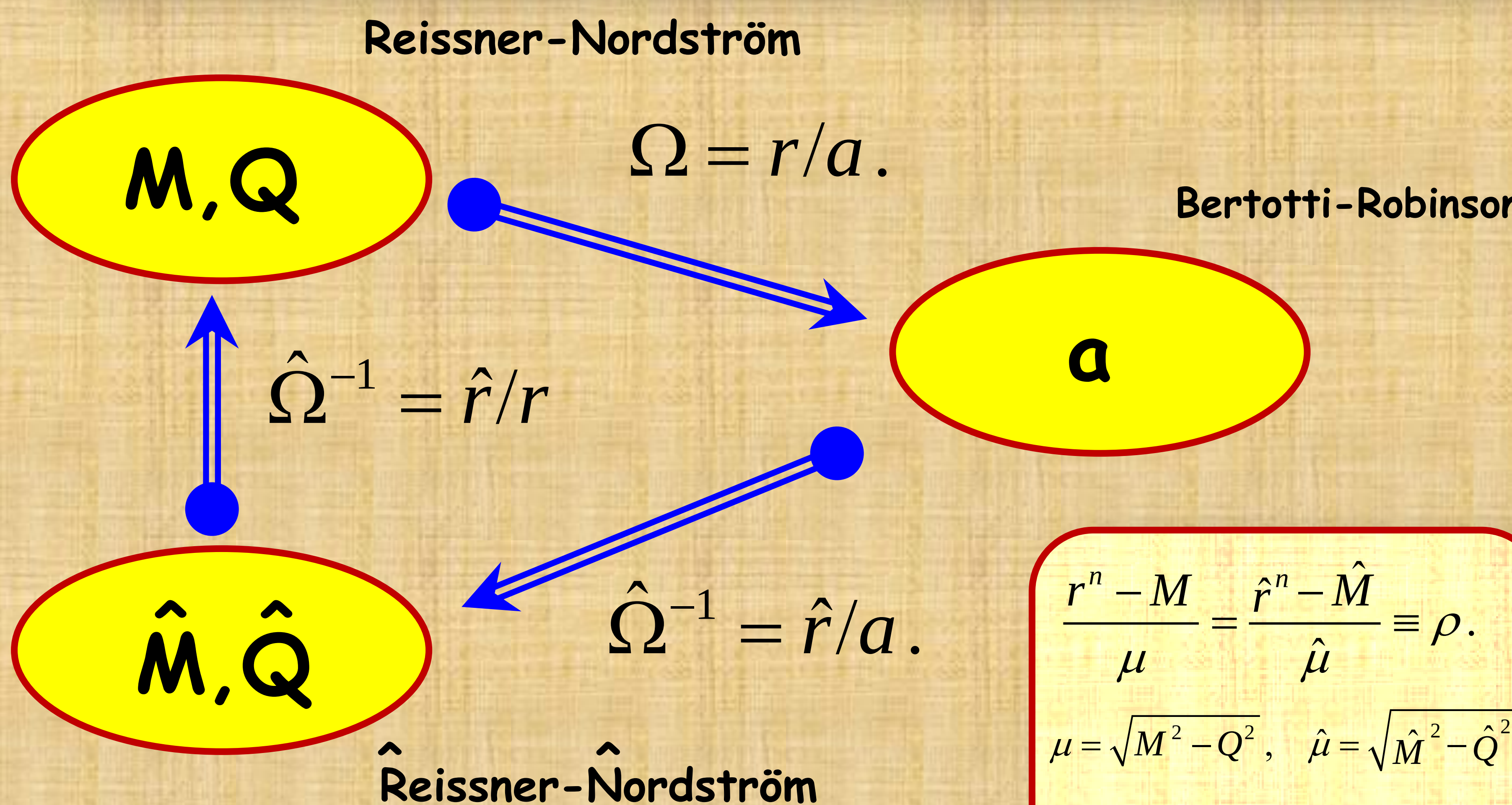
$$ds^2 = -\frac{\mu^2(\rho^2-1)}{(M+\mu\rho)^2}dt^2 + (M+\mu\rho)^{2/n}\left[\frac{1}{n^2(\rho^2-1)}d\rho^2 + d\omega_{n+1}^2\right].$$

The Bertotti-Robinson spacetime

B-R

$$d\bar{s}^2 = a^2\left[\frac{1}{n^2}\left(-(\rho^2-1)\,d\bar{\sigma}^2 + \frac{1}{\rho^2-1}d\rho^2\right) + d\omega_{n+1}^2\right],$$

Bi-conformal map between the Reissner-Nordström metrics with different M and Q



$$\frac{r^n - M}{\mu} = \frac{\hat{r}^n - \hat{M}}{\hat{\mu}} \equiv \rho\,.$$

$$\mu = \sqrt{M^2 - Q^2}\,, \quad \hat{\mu} = \sqrt{\hat{M}^2 - \hat{Q}^2}\,.$$

$$\Phi = \widehat{\Phi}\,, \quad \alpha = \Omega^{-n}\widehat{\alpha}$$

The static Green function

$$G(x,x') = \int_{-\infty}^{\infty} dt \; G_{\text{Ret}}(t,x;0,x').$$

$$\hat{F}\,G(x,x') = -\frac{\delta(x-x')}{\alpha\sqrt{g}}.$$

$$\hat{F}\,\Phi = -4\pi J\,, \qquad \hat{F} = \frac{1}{\alpha\sqrt{g}}\partial_a\left(\alpha\sqrt{g}\,g^{ab}\partial_b\right). \qquad J(x) = q\,\frac{\delta(x-y)}{\sqrt{g}}.$$

$$ds^2 = -\alpha^2 dt^2 + g_{ab}dx^a dx^b$$

The scalar potential is biconformally invariant

$$\Phi(x) = 4\pi q\,\alpha(y)\,G(x,y).$$

$$\left[n^2(\rho^2-1)\,\partial_\rho^2 + 2n^2\rho\,\partial_\rho + \Delta_\omega^{n+1}\right]G(x,x') = -\frac{n}{\mu}\delta(\rho-\rho')\delta(\omega,\omega').$$

The Green function and the heat kernel in the Bertotti-Robinson spacetime

The Euclidean Green function in D-dimensional spacetime

$$\hat{O}\, G_{\hat{O}}(X_{\mathrm{E}}, X'_{\mathrm{E}}) = -\delta^{D}(X_{\mathrm{E}}, X'_{\mathrm{E}}), \qquad \hat{O} = \Box - m^2$$

$$G_{\hat{O}}(X_{\mathrm{E}}, X'_{\mathrm{E}}) = \int_0^\infty ds\, K_{\hat{O}}(s \,|\, X_{\mathrm{E}}, X'_{\mathrm{E}}).$$

$$(\partial_s - \hat{O})\, K_{\hat{O}}(s \,|\, X_{\mathrm{E}}, X'_{\mathrm{E}}) = 0, \qquad K_{\hat{O}}(0 \,|\, X_{\mathrm{E}}, X'_{\mathrm{E}}) = \delta(X_{\mathrm{E}}, X'_{\mathrm{E}}),$$

Because the geometry of the D-dimensional Bertotti-Robinson spacetime has the form of a direct sum of two homogeneous spaces, the heat kernel K is the product of heat kernels on the hyperboloid and on the sphere

$$K_{\hat{O}}(s \,|\, \rho, \bar{\theta}_k, \theta_n; \rho', \bar{\theta}'_k, \theta'_n) = e^{-m^2 s}\, K_{H^2}(s \,|\, \rho, \bar{\theta}_k; \rho', \bar{\theta}'_k)\, K_{S^{n+1}}(s \,|\, \theta_n; \theta'_n).$$

The heat kernel on H^2

$$\cosh(\chi) = \rho\rho' - \sqrt{\rho^2 - 1}\sqrt{\rho'^2 - 1}\cos\bar{\sigma}\,.$$

$\chi =$ **geodesic distance on the hyperboloid**

$$K_{H^2}(s\,|\,\chi) = \frac{\sqrt{2}b}{(4\pi s)^{3/2}} e^{-s/(4b^2)} \int_{\chi}^{\infty} dy \frac{y\, e^{-b^2 y^2/(4s)}}{(\cosh y - \cosh(\chi))^{1/2}}\,.$$

The heat kernel on S^2

$\gamma =$ **geodesic distance on the D-sphere**

$$K_{S^2}(s\,|\,\gamma) = \frac{\sqrt{2}a}{(4\pi s)^{3/2}} e^{s/(4a^2)} \sum_{k=-\infty}^{\infty} (-1)^k \int_{\gamma}^{\pi} d\phi \frac{(\phi + 2\pi k)\, e^{-a^2(\phi+2\pi k)^2/(4s)}}{(\cos\gamma - \cos\phi)^{1/2}}\,.$$

The heat kernel on S^3

$$K_{S^3}(s\,|\,\gamma) = \frac{1}{(4\pi s)^{3/2}} e^{s/a^2} \sum_{k=-\infty}^{\infty} \frac{(\gamma + 2\pi k)\, e^{-a^2(\gamma+2\pi k)^2/(4s)}}{\sin\gamma}\,,$$

The heat kernel on S^{n+1}

$$K_{S^{n+1}}(s\,|\,\gamma) = e^{\frac{(n^2-1)s}{4a^2}} \left(\frac{1}{2\pi a^2} \frac{\partial}{\partial \cos\gamma} \right)^{\frac{(n-1)}{2}} K_{S^2}(s\,|\,\gamma), \qquad n \text{ odd},$$

$$K_{S^{n+1}}(s\,|\,\gamma) = e^{\frac{(n^2-4)s}{4a^2}} \left(\frac{1}{2\pi a^2} \frac{\partial}{\partial \cos\gamma} \right)^{\frac{(n-2)}{2}} K_{S^3}(s\,|\,\gamma), \qquad n \text{ even}.$$

The heat kernel on $H^2 \times S^{n+1}$

$$K(s\,|\,X_{\mathrm{E}}, X'_{\mathrm{E}}) = K(s\,|\,\chi, \gamma) = K_{H^2}(s\,|\,\chi) \times K_{S^{n+1}}(s\,|\,\gamma).$$

For even spacetime dimensions

$$K(s\,|\,\chi,\gamma) = -4\frac{a^2}{n}\left(\frac{1}{2\pi a^2}\frac{\partial}{\partial\cos\gamma}\right)^{(n+1)/2}\frac{1}{(4\pi s)^2}\sum_{k=-\infty}^{\infty}(-1)^k$$

$$\times\int_\chi^\infty dy\,\frac{y\,e^{-a^2y^2/(4n^2s)}}{(\cosh y-\cosh\chi)^{1/2}}\int_\gamma^\pi d\phi(\cos\gamma-\cos\phi)^{1/2}\frac{\partial}{\partial\phi}e^{-\frac{a^2(\phi+2\pi k)^2}{4s}}.$$

For odd spacetime dimensions

$$K(s\,|\,\chi,\gamma) = \frac{a}{n}\left(\frac{1}{2\pi a^2}\frac{\partial}{\partial\cos\gamma}\right)^{(n-2)/2}\frac{\sqrt{2}}{(4\pi s)^3}\frac{1}{\sin\gamma}$$

$$\sum_{k=-\infty}^{\infty}(\gamma+2\pi k)\,e^{-a^2(\gamma+2\pi k)^2/(4s)}\int_\chi^\infty dy\,\frac{y\,e^{-a^2y^2/(4n^2s)}}{(\cosh y-\cosh\chi)^{1/2}}.$$

The static Green function

$$G(\chi,\gamma) = a\int_0^{2\pi}d\sigma\int_0^\infty ds\,K(s\,|\,\chi,\gamma).$$

Here

$$\cosh(\chi) = \rho\rho' - \sqrt{\rho^2-1}\sqrt{\rho'^2-1}\cos\sigma.$$

For even spacetime dimensions D

n=D-3

$$G(x,x') = \frac{1}{n\mu}\frac{1}{2(2\pi)^{\frac{n+3}{2}}}\left(\frac{\partial}{\partial\cos\gamma}\right)^{(n+1)/2}\int_0^{2\pi} d\sigma\, A_n\,.$$

$$A_n = \int_\chi^\infty dy \frac{1}{\sqrt{\cosh(y)-\cosh(\chi)}}\frac{\sinh\left(\frac{y}{n}\right)}{\sqrt{\cosh\left(\frac{y}{n}\right)-\cos(\gamma)}}\,.$$

For odd spacetime dimensions D

$$G(x,x') = \frac{1}{n\mu}\frac{1}{\sqrt{2}\,(2\pi)^{\frac{n+4}{2}}}\left(\frac{\partial}{\partial\cos\gamma}\right)^{n/2}\int_0^{2\pi} d\sigma\, B_n\,,$$

$$B_n = \int_\chi^\infty dy \frac{1}{\sqrt{\cosh y-\cosh\chi}}\frac{\sinh\left(\frac{y}{n}\right)}{\cosh\left(\frac{y}{n}\right)-\cos\gamma}\,,$$

$$\cosh(\chi) = \rho\rho' - \sqrt{\rho^2-1}\sqrt{\rho'^2-1}\cos\sigma\,.$$

Closed form of the Green function in four dimensions

Four dimensions. D=4

$$A_1 = \ln\left(\frac{\cosh(\chi)+1}{\cosh(\chi)-\cos(\gamma)}\right).$$

$$G(x,x') = \frac{1}{4\pi\mu}\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\gamma-1+\cos^2\gamma}}$$

In terms of the Schwarzschild radial coordinate r it becomes

$$G(x,x') = \frac{1}{4\pi R},$$

$$\rho = \frac{r-M}{\sqrt{M^2-Q^2}}.$$

where

$$R^2 = (r-M)^2+(r'-M)^2-2(r-M)(r'-M)\cos\gamma-(M^2-Q^2)\sin^2\gamma.$$

It reproduces the famous result by Linet (1977)

Closed form of the Green function in five dimensions

Four dimensions. D=5

$$G(x,x') = \frac{1}{8\pi^2\mu}\frac{1}{(\rho^2-1)^{1/4}(\rho'^2-1)^{1/4}}\frac{\partial}{\partial\cos\gamma}\left\{\left[\mathbf{F}\left(\psi,\kappa\right)+\mathbf{K}\left(\kappa\right)\right]\right\},$$

where

$$\sin\psi = \cos\gamma\frac{\sqrt{2}}{\sqrt{\rho\rho' - \sqrt{\rho^2-1}\sqrt{\rho'^2-1}+1}},$$

$$\kappa = \frac{\sqrt{2}\,(\rho^2-1)^{1/4}(\rho'^2-1)^{1/4}}{\sqrt{\rho\rho' + \sqrt{\rho^2-1}\sqrt{\rho'^2-1}+1-2\cos^2\gamma}}.$$

The static Maxwell field

$$\hat{O}\,A_0 = 4\pi\, j\,, \qquad \hat{O} = \frac{1}{\alpha\sqrt{g}}\partial_a\left(\frac{1}{\alpha}\sqrt{g}\,g^{ab}\partial_b\right).$$

The equation is invariant under the *biconformal* transformations

$$A_0 = \bar{A}_0\,, \quad g_{ab} = \Omega^2\,\bar{g}_{ab}\,, \quad \alpha = \Omega^n\bar{\alpha}\,, \quad j = \Omega^{-2n-2}\,\bar{j}\,,$$

The static Green function satisfies the equation

$$\hat{O}\,G_{00}(x,x') = \frac{1}{\alpha\sqrt{g}}\delta(x-x').$$

In the Reissner-Nordström spacetime

$$ds^2 = -\frac{\mu^2(\rho^2-1)}{(M+\mu\rho)^2}dt^2 + (M+\mu\rho)^{2/n}\left[\frac{1}{n^2(\rho^2-1)}d\rho^2 + d\omega_{n+1}^2\right].$$

The equation for the static Green function becomes

$$\left[n^2\,\partial_\rho\,(M+\mu\rho)^2\,\partial_\rho + \frac{(M+\mu\rho)^2}{\rho^2-1}\Delta_\omega^{n+1}\right]G_{00}(x,x') = n\mu\,\delta(\rho-\rho')\delta(\omega,\omega').$$

The ansatz:

$$G_{00}(x,x') = -\frac{\mu^2(\rho^2-1)(\rho'^2-1)}{(M+\mu\rho)(M+\mu\rho')}H(x,x'),$$

Then the equation for H

$$\left\{n^2[(\rho^2-1)\partial_\rho^2 + 4\rho\,\partial_\rho + 2] + \Delta_\omega^{n+1}\right\}H(x,x') = -\frac{n}{\mu(\rho'^2-1)}\delta(\rho-\rho')\delta(\omega,\omega').$$

Takes the form of a Green function of a scalar massive operator $\widehat{O} = \Box - m^2$ with the mass $m^2 = -2n^2/a^2$ defined on the Bertotti-Robinson spacetime $H^4 \times S^{n+1}$

Because of the property $G_{\hat{O}}(\chi,\gamma) = -\left(\frac{n^2}{2\pi a^2}\frac{\partial}{\partial\cosh\chi}\right)\bar{G}(\chi,\gamma).$

that relates the Green function of the massive operator $\hat{O} = \Box - m^2$

and that of the scalar operator $\bar{O} = \Box$ **in (D-2)-dimensional spacetime**

$$\left\{n^2\left[(\rho^2-1)\partial_\rho^2 + 2\rho\,\partial_\rho + \frac{1}{\rho^2-1}\partial_\sigma^2\right] + \Delta_\omega^{n+1}\right\}\bar{G}(\chi,\gamma) = -\frac{n^2}{a^{n+1}}\delta(\rho-\rho')\delta(\sigma-\sigma')\delta(\omega,\omega').$$

we obtain the representation

$$G_{00} = \frac{\mu^2(\rho^2-1)(\rho'^2-1)}{(M+\mu\rho)(M+\mu\rho')}\frac{a^{n+1}}{n\mu}\int_0^{2\pi} d\sigma \sin^2\sigma \frac{\partial}{\partial\cosh\chi}\bar{G}(\chi,\gamma).$$

$$\cosh(\chi) = \rho\rho' - \sqrt{\rho^2-1}\sqrt{\rho'^2-1}\cos\sigma\,.$$

For even spacetime dimensions

$$G_{00}=\frac{\mu^2(\rho^2-1)(\rho'^2-1)}{(M+\mu\rho)(M+\mu\rho')}\frac{1}{2(2\pi)^{\frac{n+3}{2}}}\frac{1}{n\mu}\left(\frac{\partial}{\partial\cos\gamma}\right)^{(n+1)/2}\int_0^{2\pi}d\sigma\sin^2\sigma\,\tilde{A}_n,$$

$$\tilde{A}_n=\int_\chi^\infty dy\frac{1}{\sqrt{\cosh(y)-\cosh(\chi)}}\frac{\partial}{\partial y}\left[\frac{1}{\sinh(y)}\frac{\sinh\left(\frac{y}{n}\right)}{\sqrt{\cosh\left(\frac{y}{n}\right)-\cos(\gamma)}}\right].$$

For odd spacetime dimensions

$$G_{00}=\frac{\mu^2(\rho^2-1)(\rho'^2-1)}{(M+\mu\rho)(M+\mu\rho')}\frac{1}{\sqrt{2}(2\pi)^{\frac{n+4}{2}}}\frac{1}{n\mu}\left(\frac{\partial}{\partial\cos\gamma}\right)^{n/2}\int_0^{2\pi}d\sigma\sin^2\sigma\,\tilde{B}_n.$$

$$\tilde{B}_n=\int_\chi^\infty dy\frac{1}{\sqrt{\cosh y-\cosh\chi}}\frac{\partial}{\partial y}\left[\frac{1}{\sinh(y)}\frac{\sinh\left(\frac{y}{n}\right)}{\cosh\left(\frac{y}{n}\right)-\cos\gamma}\right].$$

$$\cosh(\chi)=\rho\rho'-\sqrt{\rho^2-1}\sqrt{\rho'^2-1}\cos\sigma.$$

Closed form of the Green function in four dimensions

$$G_{00}(x,x') = -\frac{1}{4\pi(M+\mu\rho)(M+\mu\rho')}\left[\mu\frac{\rho\rho'-\cos\gamma}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\gamma-\sin^2\gamma}}+M\right],$$

One should add an extra zero mode contribution $\frac{C}{4\pi(M+\mu\rho)(M+\mu\rho')}$ with a coefficient **C**, such that the flux of the electric field across any surface surrounding the charge and the black hole does not depend on the position of the charge.

$$A_0(x) = 4\pi e\, G_{00}(x,x').$$

It reproduces the famous result by Copson (1928), Linet (1977)

Closed form of the Green function in five dimensions

$$G_{00} = -\frac{1}{4\pi^2 (M+\mu\rho)(M+\mu\rho')}\left[M + \mu\frac{p_0}{q_0} - \mu\frac{\partial}{\partial\cos\gamma}W\right],$$

$$W = -q\left[\mathbf{E}(\eta,\kappa) - 2\vartheta(\cos\gamma)\mathbf{E}(\kappa)\right] + \frac{q^2+p^2}{2q}\left[\mathbf{F}(\eta,\kappa) - 2\vartheta(\cos\gamma)\mathbf{K}(\kappa)\right],$$

$$\kappa = \frac{\sqrt{q^2-p^2}}{q}, \qquad \sin\eta = \frac{q}{q_0}\,\mathrm{sign}(\cos\gamma).$$

$$p = \sqrt{\rho\rho' - \sqrt{\rho^2-1}\sqrt{\rho'^2-1} + 1 - 2\cos^2\gamma}\,/\sqrt{2}, \qquad q = \sqrt{\rho\rho' + \sqrt{\rho^2-1}\sqrt{\rho'^2-1} + 1 - 2\cos^2\gamma}\,/\sqrt{2},$$

$$p_0 = \sqrt{\rho\rho' - \sqrt{\rho^2-1}\sqrt{\rho'^2-1} + 1}\,/\sqrt{2}, \qquad q_0 = \sqrt{\rho\rho' + \sqrt{\rho^2-1}\sqrt{\rho'^2-1} + 1}\,/\sqrt{2},$$

For a point charge e, $J^{\mu} = e\,\delta(x-x')\,\delta_0^{\mu}$ **the vector potential**

$$A_0(x) = 4\pi e\,G_{00}(x,x').$$

Independently later the same result in 5D was obtained by

Taylor and Flanagan (2015)

Summary

- Using the biconformal symmetry of the field operators, we found the relation between static solutions of the scalar and the Maxwell field equation on the background of the Reissner-Nordström black hole and on the background of the homogeneous Bertotti-Robinson spacetimes.

- We obtained a useful integral representation for the scalar and electric potentials, created by point static charges in the Bertotti-Robinson spacetime and, hence, in the Reissner-Nordström spacetime too.

- In four- and five-dimensional cases we obtained the exact static Green functions in a closed form.