Thirty Years of Floer Theory for 3-manifolds

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1 Overview of the Field

Floer homology, introduced by Andreas Floer in 1988, has had a dramatic impact on the field of low-dimensional topology, particularly via its incarnations as monopole and Heegaard Floer homologies. The purpose of this workshop was to bring together both leaders in the field and rising young researchers to discuss applications and relations of Floer homology to surgery problems, the fundamental group, and quantum invariants.

Heegaard Floer homology, defined by Ozsváth-Szabó in the early 2000s, is a package of invariants for 3- and 4-dimensional manifolds, as well as for embedded knots and surfaces inside them. In its simplest form, to a closed 3-manifold Y, Heegaard Floer homology associates a graded vector-space $\widehat{HF}(Y)$. A null-homologous knot K in Y induces a filtration on the Heegaard Floer complex, and the homology of the associated graded complex is the knot Floer homology of K, which categorifies the Alexander polynomial. While the Alexander polynomial provides bounds on the knot genus and obstructions to fibering, knot Floer homology in fact detects both the knot genus and fibering. One particularly nice feature of the knot Floer complex is that it actually determines the Heegaard Floer homology of all rational surgeries along K in Y.

Heegaard Floer homology and knot Floer homology are powerful tools for understanding homology cobordism and knot concordance, as they yield a host of algebraic invariants of homology cobordism and knot concordance. A unified viewpoint from which to see that these invariants are all well-defined comes from recent work of Zemke, who shows that certain decorated cobordism maps induce particularly nice maps on the associated Heegaard Floer invariants. This viewpoint can also be promoted to the involutive Floer theory of Hendricks-Manolescu.

An interesting question is to consider how Heegaard Floer homology is related to more classical invariants such as the fundamental group. Boyer-Gordon-Watson conjectured that for a closed, irreducible, rational homology sphere Y, the following are equivalent:

1. Y is an L-space, i.e., $\widehat{HF}(Y) \cong \mathbb{Z}^{|H_1(Y)|}$.
2. $\pi_1(Y)$ does not admit a left-invariant total order (or is the trivial group).
3. Y does not have a co-oriented taut foliation.

The conjecture is true for a graph manifolds, due to Hanselman-Rasmussen-Rasmussen-Watson, as well as for large families of surgeries and branched covers of knots in S^3. Many of the tools used in the proof of the former results are interesting in their own right; for example, Hanselman-Watson have provided a reformulation of the bordered Floer theory of Lipshitz-Ozsváth-Thurston for manifolds with torus boundary using immersed curves in T^2.

In a different direction, while the Alexander polynomial has a 'quantum' definition as a Reshetikhin-Turaev-Viro invariant, and there are structural similarities between Heegaard Floer theory and categorified quantum invariants, even the combinatorial definitions of the Heegaard (monopole) Floer invariants are quite different from the categorifications of other Reshetikhin-Turaev invariants. These other categorified invariants—Khovanov homology, Khovanov-Rozansky homology, and their kin—are, at present, more closely tied to representation theory and mainstream combinatorics, but less closely tied to topology and, in particular, are less geometric than monopole Floer homology. Many researchers are actively pursuing similarities between Khovanov-Rozansky homologies and Floer-type invariants, in the hope of tying Floer-type invariants more closely to representation theory and finding extensions and applications of Khovanov-Rozansky homologies.

2 Presentation Highlights

The workshop featured three survey talks to introduce junior participants to different aspects of the field, and set the tone for the conference:

- Steven Boyer: Heegaard-Floer homology, foliations, and the left-orderability of fundamental groups.
  Abstract: In this talk we surveyed the known connections and evidence supporting the conjectured equivalence of the following three properties of a closed, connected, orientable, irreducible 3-manifold W:

  1. W admits a co-oriented taut foliation;
  2. W has a left-orderable fundamental group;
  3. W is a Heegaard-Floer L-space.

  After introducing these notions and noting that each holds when W has a positive first Betti number, we restricted our attention to the case that W is a rational homology 3-sphere, where the task of verifying whether or not any one of the properties holds is, in general, a challenge. Particular attention was paid to manifolds arising in the context of branched covers and Dehn surgery. A relativisation of the conjecture was introduced and it was described how this led, through the work of S. Boyer and A. Clay and of J. Hanselman, J. Rasmussen, S. Rasmussen and L. Watson, to the original conjecture's confirmation for graph manifolds. Throughout, emphasis was placed on discussing open problems and related conjectures.

- Josh Greene: Floer homology and Dehn surgery.
  Abstract: I will survey applications of Floer homology to Dehn surgery over the past thirty years. Slides from the talk appear on my webpage: https://www2.bc.edu/joshua-e-greene/oaxaca.pdf

- Jake Rasmussen: Floer and Khovanov, 20 years after.
  Abstract: In his seminal paper on his "Jones polynomial homology," Khovanov identifies Floer homology as one of the underlying motivations for his work. In the 20 years since then, the connections between Khovanov homology (broadly interpreted) and Floer homology have been a fruitful topic for research, but there is still much that remains mysterious. I'll describe some of our successes at understanding the relations between the two theories over the last 20 years, and speculate on where things may go in the next 20.

The remainder of the talks focused on new results by the participants:

- Sarah Rasmussen: Left orders, transverse actions, and ordered foliations.
  Abstract: I'll discuss a topological invariant associated to a left order on the fundamental group of a prime, closed, oriented 3-manifold. I'll also describe some instances in which a left order with non-vanishing invariant can be deformed to an order giving rise to a taut foliation, and I'll characterize which foliations can be achieved by such constructions.

- Ken Baker: Satellite L-space knots are braided satellites.
  Abstract: Let $\{K_n\}$ be the family of knots obtained by twisting a knot K along an unknot c. When the winding number of K about c is non-zero, we show the limit of $g(K_n)/g_4(K_n)$ is 1 if and only if the winding and wrapping numbers of K about c are equal. When equal, this leads to a description of minimal genus Seifert surfaces of K_n and eventually to a characterization of when c is a braid axis for K. We then use this characterization to show that satellite L-space knots are braided satellites (modulo a conjecture whose solution by Hanselman-Rasmussen-Watson has been announced, and sketched during this workshop). This is joint work with Kimihiko Motegi that builds upon joint work with Scott Taylor.

- Ian Zemke: TQFT structures in link Floer homology.
  Abstract: We will discuss a TQFT for the full link Floer complex, involving decorated link cobordisms. It is inspired by Juhasz's TQFT for sutured Floer homology. We will discuss how the TQFT recovers standard bounds on concordance invariants like Ozsvath and Szabo's tau invariant and Rasmussen's local h invariants (which are normally proven using surgery theory) and also gives a new bound on Upsilon. We will also see how well known maps in the link Floer complex can be encoded into decorations on surfaces, and as an example we will see how Sarkar's formula for a mapping class group action on link Floer homology is recovered by some simple pictorial relations. Time permitting, we will also discuss how these pictorial relations give a connected sum formula for Hendricks and Manolescu's involutive invariants for knot Floer homology.

- Zhongtao Wu: On Alexander polynomials of graphs.
  Abstract: Using Alexander modules, one can define a polynomial invariant for a certain class of graphs with a balanced coloring. We will give different interpretations of this polynomial by Kauffman state formula and MOY relations. Moreover, there is a Heegaard Floer homology of graphs whose Euler characteristic is this Alexander polynomial. This is joint work with Yuanyuan Bao.

- Allison Miller: Knot traces and concordance.
  Abstract: A conjecture of Akbulut and Kirby from 1978 states that the concordance class of a knot is determined by its 0-surgery. In 2015, Yasui disproved this conjecture by providing pairs of knots which have the same 0-surgeries yet which can be distinguished in (smooth) concordance by an invariant associated to the four-dimensional traces of such a surgery. In this talk, I will discuss joint work with Lisa Piccirillo in which we construct many pairs of knots which have diffeomorphic 0-surgery traces yet some of which can be distinguished in smooth concordance by the Heegaard Floer d-invariants of their double branched covers. If time permits, I will also discuss the applicability of this work to the existence of interesting invertible satellite maps on the smooth concordance group.

- Cameron Gordon: Strongly quasipositive links, cyclic branched covers, and L-spaces.
  Abstract: We give constraints on when the n-fold cyclic branched cover $\Sigma_n(L)$ of a strongly quasipositive link L can be an L-space. In particular we show that if K is a nontrivial L-space knot and $\Sigma_n(K)$ is an L-space then

  1. $n \leq 5$.
  2. If $n = 4$ or 5 then K is the torus knot $T(2,3)$.
  3. If $n = 3$ then K is either $T(2,3)$ or $T(2,5)$, or K is hyperbolic and has the same Alexander polynomial as $T(2,5)$.
  4. If $n = 2$ then $\Delta_K(t)$ is a non-trivial product of cyclotomic polynomials.

  This is joint work with Michel Boileau and Steve Boyer.

- Rachel Roberts: Alternating knots satisfy the L-space knot conjecture.
  Abstract: Co-oriented taut foliations (CTFs) are an important tool in the study of 3-manifolds; determining existence or nonexistence of CTFs in a given 3-manifold is therefore important. Heegaard Floer homology computations are currently the most effective method for establishing nonexistence, and, more recently, have been used to motivate the search for new constructions of CTFs. In this talk, I describe constructions of CTFs motivated by the Heegaard Floer computations of Ozsváth and Szabó

for surgeries along alternating knots, and those of Lidman, Baker and Moore for surgeries along Montesinos knots. It follows from these constructions that if K is an alternating knot or a Montesinos knot, then the L-space conjecture of Ozsváth and Szabó holds for any 3-manifold obtained by Dehn surgery along K. This work is joint with Charles Delman.

Given a knot is S^3, we search for a co-orientable laminar branched surface with the property that the closed complementary component containing the knot is a solid torus with a positive, even number of meridional cusps; hence, it carries a lamination whose corresponding complementary component may be filled with disk leaves after any nontrivial surgery. We currently have two methods for finding branched surfaces with these properties. The first, which may be viewed roughly as a generalization of Murasugi summing, obtains the spine of a branched surface from the boundary of a tubular neighborhood of the knot, spheres that intersect the knot transversely, and spanning surfaces for related simpler links. The second, which may be viewed as building on Gabai's taut sutured manifold hierarchy approach, obtains the spine from the boundary of a tubular neighborhood of the knot, a (not necessarily orientable) spanning surface, and decomposing disks.

This approach to constructing CTFs works more generally: whenever an irreducible 3-manifold can be realized as Dehn filling a closed 3-manifold along a knot that has a nice enough essential spanning surface (orientable or nonorientable), then it is possible to prove that the knot is *persistently foliar*; namely, that every nontrivial Dehn filling results in a 3-manifold that contains a CTF. However, we are still searching for the correct statement.

- Kristen Hendricks: Involutive Heegaard Floer homology and the homology cobordism group.
  Abstract: Involutive Heegaard Floer homology is a variant on the 3-manifold invariant Heegaard Floer homology which incorporates the data of the conjugation symmetry on the Heegaard Floer complexes, and is in principle meant to correspond to $\mathbb{Z}_4$-Seiberg Witten Floer homology. It can be used to obtain two new invariants of homology cobordism and two new concordance invariants of knots, one of which (unlike other invariants arising from Heegaard Floer homology) detects non-sliceness of the figure-eight knot. We introduce involutive Heegaard Floer homology and its associated invariants and use it to give a new criterion for an element in the integer homology cobordism group to have infinite order, similar but not identical to a recent criterion given by Lin-Ruberman-Saviliev. Much of this talk is joint work with C. Manolescu; other parts are variously also joint with I. Zemke or with J. Hom and T. Lidman.

- Linh Truong: Truncated Heegaard Floer homology and concordance invariants.
  Abstract: Heegaard Floer homology has proven to be a useful tool in the study of knot concordance. Ozsvath and Szabo first constructed the tau invariant using the hat version of Heegaard Floer homology and showed tau provides a lower bound on the slice genus. Later, Hom and Wu constructed a concordance invariant using the plus version of Heegaard Floer homology; this provides an even better lower-bound on the slice genus. In this talk, I discuss a sequence of concordance invariants that are derived from the truncated version of Heegaard Floer homology. These truncated Floer concordance invariants generalize the Ozsvath-Szabo and Hom-Wu invariants.

- Josh Greene: Fibered simple knots.
  Abstract: Simple knots in lens spaces were first studied by Berge, and they play a central role in relation to Dehn surgery and Floer homology. I will describe a characterization of which simple knots in lens spaces fiber. The answer takes a peculiar and elementary number theoretic form, and it answers a question by Cebanu. This is joint work with John Luecke.

- Jonathan Hanselman: Bordered Floer homology via immersed curves: properties and applications.
  Abstract: We will describe a new interpretation of bordered Heegaard Floer invariants in the case of a manifold M with torus boundary. In our setting, these invariants, originally defined as homotopy classes of (differential) modules over a particular algebra, take the form of homotopy classes of immersed curves in the boundary of M decorated with local systems. Moreover, pairing two bordered Floer invariants corresponds to taking the Floer homology of two sets of decorated immersed curves. In most cases this simply counts the minimal intersection number, which leads to a number of applications. This is joint work with Liam Watson and Jake Rasmussen.

- Liam Watson: Bordered Floer homology via immersed curves: the structure theorem.
  Abstract: This talk will describe some of the key components of theorem stated in the previous talk, namely, the passage from differential modules over the torus algebra to immersed curves. We give a description of type D structures in terms of train tracks in the (marked) torus, and develop a yoga for simplifying these train tracks to immersed curves with local systems. This is joint work with Jonathan Hanselman and Jake Rasmussen.

- Ciprian Manolescu: A sheaf-theoretic model for $SL(2, C)$ Floer homology.
  Abstract: I will explain the construction of a new homology theory for three-manifolds, defined using perverse sheaves on the $SL(2, C)$ character variety. Our invariant is a model for an $SL(2, C)$ version of Floer's instanton homology. The construction uses the work of Joyce and Bussi on d-critical loci and perverse sheaves of vanishing cycles associated to complex Lagrangians in a complex symplectic manifold. We have a few explicit computations for Brieskorn spheres. The new theory is related to Witten's program for a gauge theoretic interpretation of the Jones polynomial and Khovanov homology (using the Kapustin-Witten equations). This is joint work with Mohammed Abouzaid.

- Laura Starkston: Skeleta of Weinstein manifolds.
  Abstract: The Floer theory of a cotangent fiber in a symplectic cotangent bundle T^*M can be understood via the topology of the manifold M. More generally, a Weinstein manifold has a core isotropic skeleton and we can try to understand the symplectic topology of the Weinstein manifold in terms of the topology of the skeleton. Unfortunately, generically the skeleton has singularities which make its topology difficult to understand and which lead to loss of information. We will discuss a nice minimal set of singularities which we can understand combinatorially, and try to show that all Weinstein manifolds can be deformed to have a skeleton with only these nice singularities. These singularities coincide with Nadler's "arboreal singularities" where Floer theoretic calculations can locally be done combinatorially.

- Miriam Kuzbary: A new concordance group of links.
  Abstract: The knot concordance group has been the subject of much study since its introduction by Ralph Fox and John Milnor in 1966. One might hope to generalize the notion of a concordance group to links; however, the immediate generalization to the set of links up to concordance does not form a group since connected sum of links is not well-defined. In 1988, Jean Yves Le Dimet defined the string link concordance group, where a link is based by a disk and represented by embedded arcs in $D^2 \times I$. In 2012, Andrew Donald and Brendan Owens defined groups of links up to a notion of concordance based on Euler characteristic. However, both cases expand the set of links modulo concordance to larger sets and each link has many representatives in these larger groups. In this talk, I will present joint work with Matthew Hedden where we define a link concordance group based on the the knotification construction of Peter Ozsvath and Zoltan Szabo, giving a definition of a link concordance group where each link has a unique group representative. I will also present invariants for studying this group using both group theory and Heegaard-Floer Homology.

- Sucharit Sarkar: Equivariant Floer homology.
  Absrtact: Given a Lie group G acting on a symplectic manifold preserving a pair of Lagrangians setwise, I will describe a construction of G-equivariant Lagrangian Floer homology. This does not require G-equivariant transversality, which allows the construction to be flexible. Time permitting, I will talk about applying this for the $O(2)$-action on Seidel-Smith's symplectic Khovanov homology. This is joint with Kristen Hendricks and Robert Lipshitz.

3 Scientific Progress Made

Many groups of collaborators made significant progress in Oaxaca. These include projects that were started prior to the workshop, as well as collaborations that sprang out of interactions during the workshop. We highlight several of these collaborations below.

Michel Boileau, Steve Boyer, and Cameron Gordon were able to make progress on a joint project while in Oaxaca which has resulted in a submitted preprint “Branched covers of quasipositive links and L-spaces” (arXiv:1710.07658) and a paper in preparation “On definite basket links”.

Kristen Hendricks and Jennifer Hom worked out several of the remaining details needed for their short paper “A note on knot concordance and involutive knot Floer homology”, which was posted to the arXiv shortly after the conference (arXiv:1708.06839). In addition, they made progress on their project (joint with Tye Lidman) on applications of involutive Heegaard Floer homology to homology cobordism. They discussed this project with Ciprian Manolescu, which may result in further applications to homology cobordism.

Kristen Hendricks, Robert Lipshitz, and Sucharit Sarkar worked on an erratum to the paper “A flexible construction of equivariant Floer homology and applications” (arXiv:1510:02449), and discussed their next joint paper in a preliminary fashion.

Zhongtao Wu received many questions and feedback for the talk that he gave on joint work with Yuanyuan Bao, which are really helpful when they wrote up the paper a few weeks after the workshop (arXiv:1708.09092).

Marco Golla and Laura Starkston continued work on their project on symplectic rational cuspidal curves.

4 Comments from participants

Feedback from participants was overwhelmingly positive. Below are a representative selection of comments received in response to requests for feedback. (The names of the participants providing the feedback have been redacted.)

- “I thoroughly enjoyed the Oaxaca workshop. There were many interesting talks, from which I learned about new developments in the field; I was intrigued by Allison Miller’s work with Lisa Piccirillo, and by the new bordered theory for torus boundary via immersed curves, due to Hanselman-Rasmussen-Watson. I particularly liked the problem session. I also benefited from discussions with the other participants.... Also, after my talk several participants asked me follow-up questions...and they inspired some current projects that I’m working on.”
- “I loved the workshop. It is one of the very best conferences that I have attended, in terms of content, setting, organization, and the people. The problem session in particular was especially well-executed and I think is an excellent model for future problem sessions.”
- “The workshop was great. The talks were fantastic, and the conversations between were quite useful to learn about recent developments. The atmosphere was very positive and the problem session went impressively well.”
- “First of all, thanks for organizing the workshop....My first memory is that it was delightful. I felt that the organization was like clockwork, and the support staff brilliant.”
- “This was an excellent workshop; I thought the talks were well-targeted, and it was a pleasure to work in such an idyllic setting. The expository talks were extremely helpful.”
- “The Oaxaca workshop was a wonderful experience. The schedule provided ample time to meet and talk to the other participants. I enjoyed learning about recent results in Heegaard Floer theory in the excellent talks. I also had the opportunity to give a talk and I received valuable feedback from other researchers in the field.”

5 Problem Sessions

One of the key goals of this workshop was to create a list of open problems in the area. The goal was to include both structural questions related to Floer homology and questions in 3- and 4-dimensional topology which might be accessible to Floer-theoretic techniques. To this end, the conference featured two 1.5 hour problem sessions. Each problem session had a moderator, who was the only person authorized to write problems on the board and, consequently, had to understand each problem. The first problem session was moderated by William Kazez and Sarah Rasmussen, and the second by Cameron Gordon. Each session also had two scribes in the audience: Kristen Hendricks and Allison Miller for the first session and James Cornish and Ian Zemke for the second session. All workshop participants were told in advance about the problem sessions and asked to think of ideas. The organizers also asked senior participants individually to bring one or two problems.

The problem sessions were remarkably successful, and generated a large number of interesting questions. After the program, the organizers typed the problems and organized them by topic. They then sent the typed versions to the original proposers to ask for feedback, and then all participants for further feedback.

In 1977, Robion Kirby compiled a list of problems in low-dimensional topology, as part of the Georgia Topology Conference. The list was heavily revised and expanded in the mid-1990s to reflect the remarkable developments in low-dimensional topology in the 1980s and early 90s. Several topologists have begun work on a second update to the Kirby list, and we expect that the problem list from this conference will be folded into that updated list.

The problem list, in its current form, follows.

6 Problems

While we note who contributed each problem, many of the problems are well-known in certain circles and not due to the contributors. When problems have previously appeared in print, we have tried to give citations to them.

Some objects occur so frequently below that it seems preferable to collect citations to their definitions here: the Heegaard Floer invariants of closed 3-manifolds [OS04b], knot and link (Heegaard) Floer homology [OS04a, Ras03, OS08], and Khovanov homology [Kho00].

6.1 The L-space – non-left-orderable π_1 – no coorientable taut foliation conjecture

Conjecture 1. (The L-space-NLO-NCTF Conjecture) Let Y^3 be a closed, irreducible, rational homology 3-sphere. Then the following are equivalent:

1. Y is an L-space, i.e., $\widehat{HF}(Y) \cong \mathbb{Z}^{|H_1(Y)|}$.
2. $\pi_1(Y)$ does not admit a left-invariant total order (or is the trivial group).
3. Y does not have a co-oriented taut foliation.

Remarks. The conjecture is true for graph manifolds, due to work of Hanselman-Rasmussen-Rasmussen-Watson [HRRW] and Boyer-Clay [BC17]. There is also substantial computational evidence addressing parts of the conjecture arising from surgeries and branched covers of certain families of (hyperbolic) knots.

Acknowledgements. Conjecture is due to Boyer-Gordon-Watson [BGW13] and Juhász [Juh15].

Problem 2. Call a 3-manifold Y with torus boundary *Floer simple* if it has a Dehn filling whose core is a Floer simple knot. Equivalently, Y is Floer simple if Y has two different L-space fillings [RR, Proposition 1.3].

Suppose that Σ is a surface with one boundary component and $\phi : \Sigma \to \Sigma$ is a homeomorphism. Find conditions on ϕ so that the mapping torus T_ϕ is / is not Floer simple.

Remarks. R. Roberts gives classes of mapping tori with cooriented taut foliations for all but one boundary slope; these cannot be Floer simple [Rob01].

Another example is that if Y is an integer homology $D^2 \times S^1$ and Y is not fibered then Y is not Floer simple.

Note also that if Y is Floer simple then the Turaev torsion $\tau(Y)$ satisfies certain restrictions [RR].

Acknowledgements. Problem contributed by J. Rasmussen. Discussion contributed by S. Rasmussen, M. Golla.

Problem 3.

1. For a 3-manifold Y with $\partial Y = T^2$, describe the structure of the sets

$$\mathcal{D}_{fol}(Y) = \{\text{slopes } \gamma \mid Y(\gamma) \text{ admits a co-orientable taut foliation}\}$$
$$\mathcal{D}_{LO}(Y) = \{\text{slopes } \gamma \mid \pi_1(Y(\gamma)) \text{ is left-orderable}\},$$

in analogy with the Rasmussen-Rasmussen L-space filling result [RR, Theorem 1.6].

2. More generally, prove the analogues for co-oriented taut foliations and left orders of the Hanselman-Rasmussen-Watson L-space torus gluing formula [HRWa].

Remarks. Of course, these would follow from the L-NLO-NCTF Conjecture.
Acknowledgements. Problem contributed by J. Hanselman.

Problem 4. Let P be a knot in a solid torus $D^2 \times S^1$ and K a knot in S^3. Give necessary and sufficient conditions for the satellite knot $P(K)$ to be a positive L-space knot.

Remarks. Let $P_n \subset S^3$ be the result of adding n full twists to P and then including $D^2 \times S^1$ into S^3 as a neighborhood of the unknot. It is known from [Hom16, Proposition 3.3], which relies on [HRWa], that if $P(K)$ is a positive L-space knot then:

1. K is a positive L-space knot.
2. P_n is a positive L-space knot for all $n \geq -2g(K) + 1$.
3. P_n is a negative L-space knot for all $n \ll 0$.

It is also known by combining [Hom16, Theorem 1.1] and [BM, Theorem 1.9] that if the following conditions are satisfied then $P(K)$ is an L-space knot:

1. K is a positive L-space knot.
2. P_n is a positive L-space knot for all $n \geq -2g(K)$.
3. P_n is a negative L-space knot for all $n \ll 0$.

By considering cables [Hom11] [Hed09], specifically $(p, pn-1)$ and $(p, pn+1)$ cables, one can see that the first set of conditions are not sufficient to guarantee $P(K)$ is an L-space knot, and the second set of conditions are not necessary.

Baker-Motegi [BM, Theorem 7.4], relying on [HRWa], show that if $P(K)$ is an L-space knot then P is a braid.

Gillespie [Gil] gives examples that might be of interest.

Acknowledgements. Problem contributed by J. Hom, with comments by D. McCoy, J. Rasmussen.

Conjecture 5. Suppose Y_1 and Y_2 are rational homology 3-spheres.

1. If $f : Y_1 \to Y_2$ is a smooth map with degree 1 then the Heegaard Floer homology satisfies $\mathrm{rank}\widehat{HF}(Y_1) \geq \mathrm{rank}\widehat{HF}(Y_2)$.
2. More generally, if $f : Y_1 \to Y_2$ has degree $n \geq 1$ then the Heegaard Floer homology satisfies $n\mathrm{rank}\widehat{HF}(Y_1) \geq \mathrm{rank}\widehat{HF}(Y_2)$.
3. Less generally, if $f : Y_1 \to Y_2$ is a degree-1 map and Y_1 is an L-space then Y_2 is an L-space.

Remarks. In the special case that f is a regular $\mathbb{Z}/p$-covering map, the second statement was proved by Lidman-Manolescu [LM]. In the special case that the Y_i are Seifert fibered spaces, the first statement was proved by Karakurt-Lidman [KL15]. Problem 10 is also related.

The third condition would follow from the L-NLO-NCTF Conjecture and a theorem of Boyer-Rolfsen-Wiest [BRW05], since the induced map $f_* : \pi_1(Y_1) \to \pi_1(Y_2)$ is surjective.

Acknowledgements. Problem contributed by M. Miller.

Problem 6.

1. Suppose that Y is an integer homology sphere and $\pi_1(Y)$ has no nontrivial $SU(2)$-representation. Show that $Y \cong S^3$.
2. Under the same hypothesis, show that Y is an L-space. (This is weaker.)
3. If Y is a rational homology sphere so that $\pi_1(Y)$ has no irreducible $SU(2)$-representations, must Y be an L-space?
4. If Y is a rational homology sphere with left-orderable fundamental group, does $\pi_1(Y)$ admit an irreducible $SU(2)$-representation.

Remarks. Perhaps one can use an $SL(2,\mathbb{R})$-representation to construct an $SU(2)$-representation, for an integer homology sphere.

An affirmative answer to Problem 38 would resolve parts (2) and (3) of this question.

Acknowledgements. Problem contributed by M. Boileau. Remarks by M. Miller and others.

Problem 7. Construct a $\mathrm{PSL}(2,\mathbb{R})$ Floer homology for 3-manifolds and relate it to Heegaard Floer homology.

Remarks. This is a possible attack on Problem 6(2). That is, this would be a way to use certain left orders of the fundamental group to construct elements of Heegaard Floer homology. Note, however, that not every left order of $\pi_1(Y)$ comes from a $\mathrm{PSL}(2,\mathbb{R})$-representation. For example, the fundamental group of $+5$ surgery on the figure-eight knot is left orderable, but has no irreducible $\mathrm{PSL}(2,\mathbb{R})$-representations. (This last fact can be checked by an analysis similar to that in [BRW05, Example 3.13].)

The connection to Heegaard Floer homology may come from the fact that the spaces of $\mathrm{PSL}(2,\mathbb{R})$ flat connections on a Riemann surface are vector bundles over symmetric products of that surface; cf. [Hit87, theorem 10.8].

Acknowledgements. Problem contributed by C. Manolescu, with comment by L. Watson.

Problem 8. Find a 3-manifold Y with torus boundary so that the corresponding immersed curve in $T^2 \setminus \{pt\}$ [HRWa] is equipped with a nontrivial local system. *Acknowledgements.* Problem contributed by L. Watson.

Problem 9. Let $\Sigma_n(K)$ denote the n-fold cyclic cover of S^3 branched along K. Is there an $n > 5$ and a knot K so that $\Sigma_n(K)$ is an L-space but $\Sigma_{n+1}(K)$ is not an L-space?

Remarks. For $n = 5$, the trefoil is an example.

One could ask a similar question for knots in the Poincaré homology sphere.

Acknowledgements. Problem contributed by C. Gordon. Comments by K. Baker.

Problem 10. Is there a knot K and integer n so that the cyclic branched cover $\Sigma_n(K)$ is an L-space but $\Sigma_{n-1}(K)$ is not an L-space?

Acknowledgements. Problem contributed by R. Lipshitz.

6.2 Surgery problems

Problem 11.

1. Is there an integer homology 3-sphere which cannot be obtained by surgery on a 2-component link in S^3?
2. Is there an irreducible example?
3. Is there a hyperbolic example?
4. What about an n-component link for other $n > 1$?

Remarks. The presumed answer in all cases is "yes."

It is an old, open question in group theory [MK14, Problem 5.52] whether all perfect groups have weight 1, so the fundamental group is unlikely to provide an obstruction.

Acknowledgements. Problem contributed by M. Boileau and J. Greene.

Problem 12. Let $p(c)$ be the maximum number of prime components produced by surgery on some c-component link in S^3.

1. Compute $p(c)$ for some values of c (e.g., $c = 1$, $c = 2$).

2. Prove that $p(c)$ is finite for all c.

3. What is the growth rate of $p(c)$ as $c \to \infty$?

Remarks. Combining results of Sayari [Say98], Valdez-Sánchez [VS99], and Howie [How02] (see [How10]) gives that $p(1) \in \{2, 3\}$; the *Two Summands Conjecture* is the statement that $p(1) = 2$. It seems not to be known if $p(2)$ is finite.

Acknowledgements. Problem contributed by D. Auckly, with comments by C. Gordon and others.

Problem 13. Let $K \cup C$ be a 2-component link in S^3, where C is an unknot, and let K_n be the knot in S^3 obtained by performing $(-1/n)$-surgery on C. Assume:

1. K_n is strongly quasi-positive and fibered for $n \gg 0$, and

2. K_n is strongly quasi-negative and fibered for $n \ll 0$.

Using Heegaard Floer homology, show that C is a braid axis for K. *Remarks.* Fibered and strongly quasi-positive is equivalent to being the binding of an open book which supports the tight contact structure on S^3 [Hed10, Proposition 2.1].

There is a topological (non-Floer theoretic) proof by Baker-Motegi [BM], but a Heegaard Floer-theoretic proof might allow one to extend the statement to other 3-manifolds.

A possible generalization: if K is a knot which is transverse to a page of an open book, give conditions under where K braided with respect to the open book (i.e., transverse to all of the pages).

Acknowledgements. Problem contributed by K. Baker.

Problem 14. Understand the behavior of the Thurston norm of a 3-manifold with boundary a union or tori under Dehn filling of a single torus.

Remarks. Baker-Taylor show that for all but finitely many slopes, the Thurston norm, the Thurston norm in the filled manifold plus the *winding norm* of the class is equal to the Thurston norm in the un-filled manifold [BT, Theorem 4.6]. For other slopes, the Thurston norm in the filled manifold is strictly smaller than this [BT, Lemma 1.1]; Baker-Taylor call such slopes *norm-reducing*. The question is to understand the norm-reducing slopes and how much the Thurston norm drops for each.

Acknowledgements. Problem contributed by K. Baker.

Problem 15. Is every integer homology 3-sphere homology cobordant to surgery on a knot in S^3?

Acknowledgements. Problem contributed by J. Hom.

Problem 16. What can Heegaard Floer homology and related invariants say about Seifert-fibered Dehn surgeries on hyperbolic knots? Two specific conjectures are:

Conjecture 17. Every Seifert-fibered Dehn surgery on a hyperbolic knot must have integral framing coefficient.

Conjecture 18. Seifert fibered spaces with base S^2 and more than 3 singular fibers do not arise as surgery on hyperbolic knots in S^3.

Remarks. Some results about Seifert-fibered surgeries obtained so far:

- Ozsváth-Szabó gave restrictions on the Alexander polynomial and knot Floer homology of knots admitting Seifert-fibered surgeries [OS04c]. Further restrictions were obtained by M. Doig [Doi15] and and Z. Wu [Wu12].
- J. Greene solved the lens space realization problem [Gre13a].
- L. Gu solved the realization problem for T-, O-, or I-type manifolds [Gu].
- For prism manifolds (i.e., D-type manifolds) $P(p,q)$ with $q < 0$ this has been solved [BHM$^+$]. The case of $q > 0$ is in progress.
- J. Hanselman has ruled out some examples of L-space Seifert-fibered surgeries using d-invariants. (Note that, since there are non-hyperbolic counterexamples to the second conjecture, d-invariants alone cannot suffice.)

Acknowledgements. Problem contributed by M. Eudave-Muñoz. Comments by various.

Problem 19. Fix integers n, g, and d. Let $M_1 = \#^n(S^1 \times S^2)$ and let M_2 be a circle bundle over a closed surface of genus g, with Euler number d.

1. Suppose W is a cobordism from M_1 to M_2 built by attaching 2-handles along a framed link L. What can be said about L?
2. There is a canonical tight contact structure ξ_i on M_i, $i = 1, 2$. Suppose we further require that the cobordism map on Heegaard Floer homology associated to W takes $c(\xi_1)$ to $c(\xi_2)$. What further restrictions does this put on L?

Remarks. Perhaps the tools that have been used to give surgery obstructions in Heegaard Floer homology can be applied.

The case that $n = 2g$ might be interesting to consider. In this case, it is possible for L to have one component: L can be the Borromean link. Note that the Borromean link is detected by its Floer homology [Ni14].

Acknowledgements. Problem contributed by L. Starkston. Comments by Marco Golla and others.

6.3 Problems on concordance and cobordism

Problem 20.

1. Are there candidates for torsion elements in the homology cobordism groups, i.e., homology 3-spheres Y so that Y does not obviously bound a homology 4-ball but $\#^n Y$ does for some $n > 1$?
2. Is there an integer homology 3-sphere Y with an orientation-reversing self-diffeomorphism which does not bound a homology 4-ball?
3. Is there an non-slice amphichiral knot of determinant 1?

Remarks. An example for (2) gives an example for (1) with $n = 2$. An example for (3) might give an example for (2) by taking the branched double cover.

Acknowledgements. Problem contributed by C. Manolescu.

Problem 21. Consider the subgroup of the smooth concordance group generated by knots with Alexander polynomial 1. Does this group contain any 2-torsion?

Remarks. This problem was proposed by Hedden-Kim-Livingston [HKL16]. It would give an affirmative answer to Problem 20(3).

Acknowledgements. Problem contributed by S. Wang.

Problem 22. Does the 3-dimensional homology cobordism group have a $\mathbb{Z}^2$ summand? A $\mathbb{Z}^\infty$ summand?

Acknowledgements. Problem contributed by S. Sarkar.

Problem 23.

1. Is there an integer homology 3-sphere Y which bounds an integer homology 4-ball but does not embed in S^4?

2. Is there an integer homology 3-sphere Y which embeds in $S^2 \times S^2$ but does not embed in S^4?

Remarks. An affirmative answer to the second question implies an affirmative answer to the first, as any $Y \subset S^2 \times S^2$ bounds an integer homology 4-ball.

Acknowledgements. Problems contributed by M. Miller and M. Golla.

Problem 24. Is there an irreducible integer homology 3-sphere Y which bounds an integer homology 4-ball but does not bound a contractible 4-manifold?

Remarks. There are reducible examples, such as $Y\#(-Y)$ where Y bounds a 4-manifold with nonstandard, definite intersection form. (Taubes attributes this observation to Akbulut [Tau87, Proposition 1.7].)

Auckly proposes that composing a reducible example with an invertible cobordism to a hyperbolic 3-manifold should give an irreducible example, using existing technology.

Acknowledgements. Problem contributed by M. Golla. Comments by D. Auckly and others.

Problem 25. Are there interesting invariants giving lower bounds for $g_4(K)$ that can give better bounds than the signature for alternating knots, but are not also lower bounds for $g_4^{\text{top}}(K)$?

In particular, are there invariants capable of showing that there are alternating knots with $g_4(K) > g_4^{\text{top}}(K) + 1$?

Remarks. L. Lewark has shown that if ν is *slice-torus* invariant, then $\nu(K) = \frac{\sigma(K)}{2}$ whenever K is alternating. Here, ν is a slice-torus invariant if it is a homomorphism from the concordance group to the real numbers with $|\nu| \leq g_4$ and $\nu(K) = g_4(K)$ whenever K is a positive torus knot [Lew14].

One can use Donaldson's theorem to show that $g_4(K) \geq g_4^{\text{top}}(K) + 1$ sometimes occurs for alternating knots.

Acknowledgements. Problem contributed by D. McCoy, with comments by A. Miller and others.

Problem 26. Let K_0 and K_1 be knots in S^3 and Σ a ribbon concordance from K_1 to K_0.

1. Is the induced map $\Phi_\Sigma : \widehat{HFK}(K_0) \to \widehat{HFK}(K_1)$ injective?

2. Weaker, is there a rank inequality between $\widehat{HFK}(K_0)$ and $\widehat{HFK}(K_1)$?

3. Is there an inequality of Seifert genera $g(K_0) \leq g(K_1)$?

4. Is the Seifert genus super-additive under band sum of > 2 knots?

Remarks. For (4), note that Gabai showed that the genus is super-additive under band sum of 2 knots [Gab87].

Acknowledgements. Problem contributed by S. Sarkar.

6.4 Other purely topological problems

Problem 27. Is there an alternating, quasi-positive link which is not strongly quasi-positive?

Acknowledgements. Problem contributed by M. Boileau.

Problem 28. What is a positive knot?

Remarks. The questions should be interpreted in the sense of R. Fox's question, recently answered by J. Greene [Gre] and J. Howie [How17], "what is an alternating knot?" That is, one would like a diagram-independent definition of positive or quasi-positive. One precise version would be to give an algorithm to determine from a diagram if the corresponding knot is positive.

Note that there is a 4-dimensional characterization of *quasi-positive* knots, due to Rudolph [Rud83] and Boileau-Orevkov [BO01]. It would be interesting to have a 3-dimensional characterization of quasi-positive knots, however, perhaps in terms of contact topology.

Acknowledgements. Problem contributed by J. Greene, with comments by M. Golla.

Problem 29.

1. Which integer homology 3-spheres admit taut foliations with non-zero Godbillon-Vey invariants?
2. Which integer homology sphere graph manifolds have non-zero Seifert volume?

Remarks. Note that there are hyperbolic integer homology 3-spheres with Seifert volume 0 [BG84].
Acknowledgements. Problem contributed by M. Boileau.

Problem 30. For $K \subset Y^3$ nullhomologous with $Y \setminus K$ hyperbolic, compare the mapping class group of the pair (Y, K) with the mapping class group of $Y_r(K)$. In particular, call a framing r *exceptional* if the two mapping class groups differ. What can be said about the set of exceptional framings r?

Remarks. For r sufficiently large, the two groups coincide. One could try to obtain results by an analogue of Thurston's exceptional surgery program for hyperbolic knots.

See [HW94, PP09] for results on this problem. An undergraduate student of Boyer's did some further experimental work along these lines; exceptional framings seemed to be quite rare [Boy].

Acknowledgements. Problem contributed by D. Auckly, with comments by S. Boyer.

Problem 31.

1. When is a Hopf basket [Rud01] a positive braid?
2. Are all non-cable L-space knots positive braids?

Remarks. Note that the $(2, 3)$-cable of the right-handed trefoil is not a positive braid; it is not even a positive knot. This follows from a result of Cromwell [Cro89], which says that for a fibered positive knot K, the genus is bounded below by $c/4$, where c is crossing number. But the genus of $(2, 3)$-cable of the right-handed trefoil is 3, while the crossing number is at least 13.

Acknowledgements. Problem contributed by K. Baker.

6.5 Internal problems to low-dimensional Floer theories

Conjecture 32. (The *Heegaard Floer Poincaré conjecture*). If a closed, irreducible 3-manifold Y has $H_1(Y) = 0$ and $\widehat{HF}(Y) \cong \mathbb{Z}$ then Y is homeomorphic to either S^3 or the Poincaré homology sphere (with either orientation).

Remarks. Combined with the L-space-NLO-NCTF Conjecture, this would imply the usual Poincaré conjecture.

The theorem is true for manifolds admitting a JSJ torus by work of Eftekhary [Eft] (see also [HL16, HRWb]).

Acknowledgements. Problem is due to Ozsváth-Szabo [OS06].

Problem 33. What polynomials occur as the Alexander polynomials of L-space knots?

Remarks. All restrictions currently known are found in a paper of Krcatovich [Krc].

Acknowledgements. Problem contributed by K. Baker.

Problem 34.

1. Is there a knot (or link) K in S^3 so that the knot Floer homology $\widehat{HFK}(K)$ is not a free abelian group?
2. Is there a rational homology sphere Y so that the Heegaard Floer homology $\widehat{HF}(Y)$ is not a free abelian group?

Remarks. Jabuka-Mark have shown that there is torsion in $HF^{\infty}(\Sigma_g \times S^1)$ for Σ_g a closed surface of genus $g \geq 3$ [JM08].

Acknowledgements. Problem contributed by R. Lipshitz.

Problem 35. Extra gradings on Heegaard Floer homology:

1. Consider the Ozsváth-Szabó spectral sequence $\widetilde{Kh}(m(K)) \Rightarrow \widehat{HF}(\Sigma(K))$ [OS05]. Do the differentials preserve the grading $\delta = i - j/2$, where i (respectively j) is the homological (respectively quantum) grading on $\widetilde{Kh}$?
2. More generally, are there other as-yet undiscovered gradings on the Heegaard Floer homology of closed 3-manifolds?
3. Can these extra gradings be used to obtain homology cobordism invariants, along the lines of the Υ invariant for knots?

Remarks. The first question is due to Greene [Gre13b]. The Szabó spectral sequence [Sza15] is δ-graded, so a proof of Conjecture 50 would imply a positive answer.

Acknowledgements. Problem contributed by B. Wong.

Problem 36. Define $f : \mathbb{Z}_{\geq 0} \to \mathbb{N}$ by

$$f(n) = \min\{\dim \widehat{HF}(Y) \mid Y \text{ has } n \text{ JSJ tori}\}.$$

Describe the properties of the map f.

Remarks. The known values of f are $f(0) = 1$ (from $Y = S^3$) and $f(1) = 5$ (from [HRWa]).

Acknowledgements. Problem contributed by L. Watson.

Problem 37. Call a 2-component link $L = A \cup B$ *exchange braided* if A and B are unknots, A is a braid axis for B, and B is a braid axis for A. In this case, let B_n denote the image of B under $1/n$-surgery on A. Are there exchange braided links so that $\{B_n\}$ contains only finitely many L-space knots?

Remarks. If $L = A \cup B$ is a 2-component link in which A and B are unknots, if $\{B_n\}$ contains infinitely many L-space knots then $A \cup B$ is exchange braided [BM].

Acknowledgements. Problem contributed by K. Baker.

Problem 38.

1. For any rational homology sphere Y, is there an isomorphism of $\mathbb{F}_2$-vector spaces $I^\#(Y) \cong \widehat{HF}(Y)$? More generally, for a sutured manifold (Y, Γ) is there an isomorphism $I^\#(Y, \Gamma) \cong SFH(Y, \Gamma)$?
2. Is Frøyshov's invariant from instanton Floer homology [Fy02] equal to the d-invariant from Heegaard Floer homology [OS03].

Remarks. The result is known for double branched covers over quasi-alternating knots, for formal reasons: both invariants agree with Khovanov homology for that class. Few other computations are known, so a first step might be computing some more examples in the instanton setting.

Acknowledgements. Problems contributed by M. Miller and C. Manolescu; originally proposed by Kronheimer-Mrowka [KM10, Conjecture 7.24]; see their paper for further conjectures and discussion along these lines. Comments by R. Lipshitz, M. Miller, and C. Scaduto.

Problem 39.

1. Is there a property of Heegaard Floer homology which would imply that the Heegaard genus of Y is at least 3?
2. Can knot Floer homology be used to bound the tunnel number of a knot?

Remarks. There are papers using TQFT techniques to give bounds on the Heegaard genus [Gar98] and tunnel number [Koh94], though the techniques do not seem to carry over to Heegaard Floer homology.

One could also ask about using Heegaard Floer homology to bound the genus of trisections.

Acknowledgements. Problem contributed by K. Baker, with comments by L. Starkston, R. Lipshitz, and others.

Problem 40. Call a 3-manifold Y with torus boundary *Floer simple* if Y has more than 1 L-space filling. What is the minimal fiber genus of a Floer simple hyperbolic integer homology $S^1 \times D^2$? A specific conjecture is:

Conjecture 41. The minimum fiber genus is 5, arising from $S^3 \setminus \mathrm{nbd}(P(-2,3,7))$.

Remarks. Note that Floer simple, hyperbolic manifolds are fibered.

Acknowledgements. Problem contributed by J. Rasmussen.

Problem 42. Call a nullhomologous knot $K \subset Y$ *Floer minimal* if $\mathrm{rank}\widehat{HFK}(Y,K) = \mathrm{rank}\widehat{HF}(Y)$.

1. *Conjecture.* Suppose $K_1, K_2 \subset Y$ are Floer minimal, K_1 is fibered, and $[K_1] = n[K_2] \in H_1(Y)$ for some $n \in \mathbb{Z}$, $n \neq 0$. Must K_2 be fibered?
2. Are there good ways to construct Floer minimal knots in non-L-spaces?
3. Does every $Y \neq S^3$ admit such a nontrivial Floer minimal knot?

Remarks. The first conjecture is true in S^3 (since the only Floer minimal knot is the unknot) and in lens spaces (Greene-Luecke, *Fibered simple knots*, work in progress).

Acknowledgements. Problems contributed by J. Greene, K. Baker.

Problem 43. Is there a 3-manifold Y and a Heegaard diagram $\mathcal{H}$ for Y so that:

- Y is not an L-space but
- the differential on the Heegaard Floer complex $\widehat{CF}(Y)$ vanishes?

(This is a Floer-theoretic analogue of a perfect Morse function.) If so, how common is this?

Acknowledgements. Problem contributed by J. Rasmussen.

Problem 44. Let K be a non-trivial knot obtained as a band sum of two unknots, and let K_n be the knot obtained by adding n full twists to the band, so that $K_0 = K$. These knots have $\widehat{HFK}(K_n)$ all isomorphic but are distinguished by Khovanov homology [HW]. Can they be distinguished in a reasonable way by some other invariant?

Acknowledgements. Problem contributed by L. Watson.

Problem 45. Show that Heegaard Floer homology defines a $(\infty, 1)$-functor from an appropriate cobordism category to the category of chain complexes.

Remarks. This would be useful for a number of applications, including extending involutive Heegaard Floer homology to a $\mathrm{Pin}(2)$-equivariant Heegaard Floer homology.

One step would be to construct a contractible CW complex with vertices Heegaard diagrams for Y and 1-cells perhaps given by certain handleslides and (de)stabilizations, on which the diffeomorphism group of Y acts. An approach to doing so might use Igusa's framed functions [Igu87, Lur09, EM12].

It might be easier to solve the analogous problem for monopole Floer homology, which would also be interesting and perhaps useful.

Acknowledgements. Problem contributed by S. Sarkar. Comments by R. Lipshitz and others.

Problem 46. Compute some $PU(n)$ "Casson invariants" of 3-manifolds with embedded links (possibly empty), when there are no reducible representations. These are the Euler characteristics of higher rank instanton homology theories for "admissible bundles" as defined by Kronheimer and Mrowka.

Acknowledgements. Problem contributed by C. Scaduto.

Problem 47. Are there finitely many "integer basic class" type invariants which determine the $U_q(\mathfrak{sl}_n)$ level k WRT (Witten-Reshetikhin-Turaev) invariants of 3-manifolds?

Alternate statement:

Problem 48. In the physics literature, Gukov, Putrov and Vafa [GPV16] constructed some unified quantum invariants with integer coefficients for 3-manifolds, which determine all the $U_q(\mathfrak{sl}_n)$ level k WRT (Witten-Reshetikhin-Turaev) invariants. Give a mathematically rigorous definition of the Gukov-Putrov-Vafa invariants.

Remarks. For integral homology spheres and $n = 2$, the WRT invariants can be expressed in terms of the Habiro power series, cf. [Hab08]. This result was extended to some rational homology spheres by Le [Lê08] and Beliakova-Blanchet-Le, cf. [BBL08].

Perhaps the Gukov-Putrov-Vafa invariants would be determined as the counts of solutions of certain differential equations: the Kapustin-Witten equations with Nahm poles, as in [Wit12].

Acknowledgements. Problem contributed by D. Auckly, with comments by C. Manolescu.

Problem 49. Let K be a nullhomologous knot in a rational homology sphere Y, say, and $\Sigma_n(K)$ the n-fold cyclic cover of Y branched along K. Let $\widetilde{K} \subset \Sigma_n(K)$ denote the branch set, i.e., the preimage of K.

1. What is the growth rate of $\dim \widehat{HF}(\Sigma_n(K))$ as $n \to \infty$? Of $\dim HF_{\mathrm{red}}(\Sigma_n(K))$?

2. What is the growth rate of $\dim \widehat{HFK}(\Sigma_n(K), \widetilde{K}, j)$ as $n \to \infty$? Of

$$\dim HFK_{\mathrm{red}}(\Sigma_n(K), \widetilde{K}, j) = \dim \operatorname{coker}\big(U^N : HFK^-(\Sigma_n(K), \widetilde{K}) \to HFK^-(\Sigma_n(K), \widetilde{K}, j)\big)$$

   for $N \gg 0$, as $n \to \infty$? Here, j denotes the Alexander grading.

3. Suppose that K is fibered with fiber F and monodromy ϕ. Fix a pointed matched circle representing F. Then there is a bordered Heegaard Floer bimodule $\widehat{CFDA}(\phi)$ associated to the map ϕ, which decomposes along Alexander gradings (spin^c-structures on F) as

$$\widehat{CFDA}(\phi) = \bigoplus_{i=-g(F)}^{g(f)} \widehat{CFDA}(\phi, i).$$

   What is the growth rate of

$$\dim H_* \widehat{CFDA}(\phi, i)^{\boxtimes_{\mathcal{A}(F,i)} n}$$

   as $n \to \infty$?

4. Let ϕ be a mapping class of a closed surface F and T_ϕ the mapping torus of ϕ. Given $i \in \mathbb{Z}$, let

$$\widehat{HF}(T_\phi, i) = \bigoplus_{\substack{\mathfrak{s} \in \mathrm{spin}^c(Y) \\ \langle c_1(\mathfrak{s}), [F] \rangle = 2i}} \widehat{HF}(T_\phi, \mathfrak{s})$$

$$HF_{\mathrm{red}}(T_\phi, i) = \bigoplus_{\substack{\mathfrak{s} \in \mathrm{spin}^c(Y) \\ \langle c_1(\mathfrak{s}), [F] \rangle = 2i}} HF_{\mathrm{red}}(T_\phi, \mathfrak{s}).$$

   What are the growth rates of $\dim \widehat{HF}(T_{\phi^n}, i)$ and $\dim HF_{\mathrm{red}}(T_{\phi^n}, i)$?

Remarks. The torsion subgroup of $H_1(\Sigma_n(K))$ grows as μ^n where μ is the Mähler measure of the Alexander polynomial $\Delta_K(t)$ [GAnS91, SW02], so the dimension of $\widehat{HF}(\Sigma_n(K))$ grows at least this fast. By contrast, Hedden-Mark [HM] show that for fibered knots with pseudo-Anosov monodromy and nonzero fractional Dehn twist coefficient, $\widehat{HF}_{\mathrm{red}}(\Sigma_n(K))$ grows at least linearly.

With notation as in point (3), there are inequalities

$$\dim H_* \widehat{CFDA}(\phi, j)^{\boxtimes_{\mathcal{A}(F,j)} n} \leq \dim \widehat{HFK}(\Sigma_n(K), \widetilde{K}, j)$$

$$\dim \widehat{HF}(\Sigma_n(K)) \leq \sum_j \dim \widehat{HFK}(\Sigma_n(K), \widetilde{K}, j).$$

In point (3), for an appropriate choice of pointed matched circle there are isomorphisms of vector spaces

$$H_* \widehat{CFDA}(\phi, -i)^{\boxtimes_{\mathcal{A}(F,-i)} n} \cong H_* \widehat{CFDA}(\phi, i)^{\boxtimes_{\mathcal{A}(F,i)} n}$$
$$H_* \widehat{CFDA}(\phi, -g(F))^{\boxtimes_{\mathcal{A}(F,-g(F))} n} \cong \mathbb{F}_2$$

Lipshitz-Ozsváth-Thurston showed that for ψ pseudo-Anosov,

$$\lim_{n\to\infty} \left(\dim H_* \widehat{CFDA}(\phi, -i)^{\boxtimes_{\mathcal{A}(F,-i)} n}\right)^{1/n} = \lambda,$$

the dilatation of ϕ [LOT13]. More generally, J. Cornish notes that if ϕ is pseudo-Anosov then the growth rate in (3) is at most $\lambda^{g(F)}$.

Dimitrov-Haiden-Katzarkov-Kontsevich call the growth rate in (3) the *entropy* of the functor $\widehat{CFDA}(\phi, i)\boxtimes \cdot$ [DHKK14] (see also [Aur10]).

Spano has given a conjectural interpretation of knot Floer homology via an analogue of embedded contact homology. His conjecture implies that if K is fibered with genus g fiber and pseudo-Anosov monodromy then the dimension of $\widehat{HFK}(\Sigma^n(K), \widetilde{K}, -g+1)$ grows exponentially in the dilatation [Spa14, p. 92].

Note that
$$\dim \widehat{HF}(T_{\phi^n}, -g(F)+1) = 1.$$

Cornish has shown that
$$\lim_{n\to\infty} \left(\dim \widehat{HF}(T_{\phi^n}, -g(F)+2)\right)^{1/n} = \lambda,$$

the dilatation of ϕ.

One can also ask the analogue of Question (3) for non-fibered knots, and the analogue of Question 4 for arbitrary self-cobordisms of a surface.

Acknowledgements. Problem contributed by R. Lipshitz. Comments by J. Cornish.

6.6 Relations with quantum invariants and categorification

Conjecture 50. Szabó's "geometric spectral sequence" [Sza15] agrees with the Ozsváth-Szabó spectral sequence $\widetilde{Kh}(m(K)) \Rightarrow \widehat{HF}(\Sigma(K))$ [OS05]. That is, for each $n \geq 1$ there is an isomorphism between the E^n-pages of the two spectral sequences. In particular, $\widehat{HF}(\Sigma(K))$ is isomorphic, as ungraded $\mathbb{F}_2$-vector spaces, to the E^∞-page of Szabó's spectral sequence.

Remarks. The conjecture has been verified for a large number of knots by programs of C. Seed and B. Zhan. The conjecture would imply the existence of an extra grading, the *δ-grading* on the Ozsváth-Szabó spectral sequence and, in particular, $\widehat{HF}(\Sigma(K))$.

Acknowledgements. Problem due to Z. Szabó [Sza15].

Conjecture 51. Let K be a knot in S^3.

1. There is a spectral sequence from the Khovanov-Rozansky HOMFLY-PT homology of K [KR08] to the knot (Heegaard) Floer homology of K.

2. There is a spectral sequence from the reduced Khovanov homology of K to the knot Floer homology of K.

3. In particular, there a rank inequality between the Khovanov and Khovanov-Rozansky homologies of K and the knot Floer homology of K.

Remarks. The first conjecture is due to Dunfield-Gukov-Rasmussen [DGR06]. The second is due to Kronheimer-Mrowka [KM10].

Rasmussen constructed a spectral sequence from Khovanov-Rozansky HOMFLY-PT homology to $\mathfrak{sl}(N)$ Khovanov-Rozansky homology for each $N > 0$ [Ras15]. It is folklore that his techniques also give a spectral sequence from $\mathfrak{sl}(N)$ homology to $\mathfrak{sl}(M)$ homology for any $N > M \in \mathbb{N}$ [Web13]. Philosophically, knot Floer homology corresponds, in some sense, to $\mathfrak{sl}(0)$.

There is work towards constructing a spectral sequence from HOMFLY-PT homology to Heegaard Floer homology by C. Manolescu [Man14] and N. Dowlin [Dow]. There is work towards constructing a spectral sequence from (a variant of) Khovanov homology to knot Floer homology by Baldwin-Levine [BL12] and Baldwin-Levine-Sarkar [BLS17].

Acknowledgements. Problem contributed by J. Rasmussen.

References

[Aur10] Denis Auroux. Fukaya categories of symmetric products and bordered Heegaard-Floer homology. *J. Gökova Geom. Topol. GGT*, 4:1–54, 2010.

[BBL08] Anna Beliakova, Christian Blanchet, and Thang T. Q. Lê. Unified quantum invariants and their refinements for homology 3-spheres with 2-torsion. *Fund. Math.*, 201(3):217–239, 2008.

[BC17] Steven Boyer and Adam Clay. Foliations, orders, representations, L-spaces and graph manifolds. *Adv. Math.*, 310:159–234, 2017.

[BG84] Robert Brooks and William Goldman. Volumes in Seifert space. *Duke Math. J.*, 51(3):529–545, 1984.

[BGW13] Steven Boyer, Cameron McA. Gordon, and Liam Watson. On L-spaces and left-orderable fundamental groups. *Math. Ann.*, 356(4):1213–1245, 2013.

[BHM$^+$] William Ballinger, Chloe Ching-Yun Hsu, Wyatt Mackey, Yi Ni, Tynan Ochse, and Faramarz Vafaee. The prism manifold realization problem. arXiv:1612.04921.

[BL12] John A. Baldwin and Adam Simon Levine. A combinatorial spanning tree model for knot Floer homology. *Adv. Math.*, 231(3-4):1886–1939, 2012.

[BLS17] John A. Baldwin, Adam Simon Levine, and Sucharit Sarkar. Khovanov homology and knot Floer homology for pointed links. *J. Knot Theory Ramifications*, 26(2):1740004, 49, 2017.

[BM] Kenneth L. Baker and Kimihiko Motegi. Seifert vs slice genera of knots in twist families and a characterization of braid axes. arXiv:1705.10373.

[BO01] Michel Boileau and Stepan Orevkov. Quasi-positivité d'une courbe analytique dans une boule pseudo-convexe. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(9):825–830, 2001.

[Boy] Steven Boyer. Personal communication.

[BRW05] Steven Boyer, Dale Rolfsen, and Bert Wiest. Orderable 3-manifold groups. *Ann. Inst. Fourier (Grenoble)*, 55(1):243–288, 2005.

[BT] Kenneth L. Baker and Scott A. Taylor. Dehn filling and the thurston norm. arXiv:1608.02443.

[Cro89] P. R. Cromwell. Homogeneous links. *J. London Math. Soc. (2)*, 39(3):535–552, 1989.

[DGR06] Nathan M. Dunfield, Sergei Gukov, and Jacob Rasmussen. The superpolynomial for knot homologies. *Experiment. Math.*, 15(2):129–159, 2006.

[DHKK14] G. Dimitrov, F. Haiden, L. Katzarkov, and M. Kontsevich. Dynamical systems and categories. In *The influence of Solomon Lefschetz in geometry and topology*, volume 621 of *Contemp. Math.*, pages 133–170. Amer. Math. Soc., Providence, RI, 2014.

[Doi15] Margaret I. Doig. Finite knot surgeries and Heegaard Floer homology. *Algebr. Geom. Topol.*, 15(2):667–690, 2015.

[Dow] Nathan Dowlin. A categorification of the HOMFLY-PT polynomial with a spectral sequence to knot Floer homology. arXiv:1703.01401.

[Eft] Eaman Eftekhary. Bordered Floer homology and existence of incompressible tori in homology spheres. arXiv:1504.05329.

[EM12] Y. M. Eliashberg and N. M. Mishachev. The space of framed functions is contractible. In *Essays in mathematics and its applications*, pages 81–109. Springer, Heidelberg, 2012.

[Fy02] Kim A. Frø yshov. Equivariant aspects of Yang-Mills Floer theory. *Topology*, 41(3):525–552, 2002.

[Gab87] David Gabai. Genus is superadditive under band connected sum. *Topology*, 26(2):209–210, 1987.

[GAnS91] Francisco González-Acuña and Hamish Short. Cyclic branched coverings of knots and homology spheres. *Rev. Mat. Univ. Complut. Madrid*, 4(1):97–120, 1991.

[Gar98] Stavros Garoufalidis. Applications of quantum invariants in low-dimensional topology. *Topology*, 37(1):219–224, 1998.

[Gil] Thomas J. Gillespie. L-space fillings and generalized solid tori. arXiv:1603.05016.

[GPV16] Sergei Gukov, Pavel Putrov, and Cumrun Vafa. Fivebranes and 3-manifold homology. arxiv:1602.05302, 2016.

[Gre] Joshua Evan Greene. Alternating links and definite surfaces. arXiv:1511.06329.

[Gre13a] Joshua Evan Greene. The lens space realization problem. *Ann. of Math. (2)*, 177(2):449–511, 2013.

[Gre13b] Joshua Evan Greene. A spanning tree model for the Heegaard Floer homology of a branched double-cover. *J. Topol.*, 6(2):525–567, 2013.

[Gu] Liling Gu. Integral finite surgeries on knots in S^3. arXiv:1401.6708.

[Hab08] Kazuo Habiro. A unified Witten-Reshetikhin-Turaev invariant for integral homology spheres. *Invent. Math.*, 171(1):1–81, 2008.

[Hed09] Matthew Hedden. On knot Floer homology and cabling. II. *Int. Math. Res. Not. IMRN*, (12):2248–2274, 2009.

[Hed10] Matthew Hedden. Notions of positivity and the Ozsváth-Szabó concordance invariant. *J. Knot Theory Ramifications*, 19(5):617–629, 2010.

[Hit87] N.J. Hitchin. The self-duality equations on a Riemann surface. *Proc. Lond. Math. Soc.*, 55:59–126, 1987.

[HKL16] Matthew Hedden, Se-Goo Kim, and Charles Livingston. Topologically slice knots of smooth concordance order two. *J. Differential Geom.*, 102(3):353–393, 2016.

[HL16] Matthew Hedden and Adam Simon Levine. Splicing knot complements and bordered Floer homology. *J. Reine Angew. Math.*, 720:129–154, 2016.

[HM] Matthew Hedden and Thomas E. Mark. Floer homology and fractional Dehn twists. arXiv:1501.01284.

[Hom11] Jennifer Hom. A note on cabling and L-space surgeries. *Algebr. Geom. Topol.*, 11(1):219–223, 2011.

[Hom16] Jennifer Hom. Satellite knots and L-space surgeries. *Bull. Lond. Math. Soc.*, 48(5):771–778, 2016.

[How02] James Howie. A proof of the Scott-Wiegold conjecture on free products of cyclic groups. *J. Pure Appl. Algebra*, 173(2):167–176, 2002.

[How10] James Howie. Can Dehn surgery yield three connected summands? *Groups Geom. Dyn.*, 4(4):785–797, 2010.

[How17] Joshua A. Howie. A characterisation of alternating knot exteriors. *Geom. Topol.*, 21(4):2353–2371, 2017.

[HRRW] Jonathan Hanselman, Jacob Rasmussen, Sarah Dean Rasmussen, and Liam Watson. Taut foliations on graph manifolds. arXiv:1508.05911.

[HRWa] Jonathan Hanselman, Jacob Rasmussen, and Liam Watson. Bordered floer homology for manifolds with torus boundary via immersed curves. In preparation.

[HRWb] Jonathan Hanselman, Jacob Rasmussen, and Liam Watson. Bordered Floer homology for manifolds with torus boundary via immersed curves. arXiv:1604.03466.

[HW] Matthew Hedden and Liam Watson. On the geography and botany of knot Floer homology. arXiv:1404.6913, to appear in Selecta Mathematica.

[HW94] Craig D. Hodgson and Jeffrey R. Weeks. Symmetries, isometries and length spectra of closed hyperbolic three-manifolds. *Experiment. Math.*, 3(4):261–274, 1994.

[Igu87] Kiyoshi Igusa. The space of framed functions. *Trans. Amer. Math. Soc.*, 301(2):431–477, 1987.

[JM08] Stanislav Jabuka and Thomas E. Mark. On the Heegaard Floer homology of a surface times a circle. *Adv. Math.*, 218(3):728–761, 2008.

[Juh15] András Juhász. A survey of Heegaard Floer homology. In *New ideas in low dimensional topology*, volume 56 of *Ser. Knots Everything*, pages 237–296. World Sci. Publ., Hackensack, NJ, 2015.

[Kho00] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[KL15] Ça˘ grıKarakurt and Tye Lidman. Rank inequalities for the Heegaard Floer homology of Seifert homology spheres. *Trans. Amer. Math. Soc.*, 367(10):7291–7322, 2015.

[KM10] Peter Kronheimer and Tomasz Mrowka. Knots, sutures, and excision. *J. Differential Geom.*, 84(2):301–364, 2010.

[Koh94] Toshitake Kohno. Topological invariants for 3-manifolds using representations of mapping class groups. II. Estimating tunnel number of knots. In *Mathematical aspects of conformal and topological field theories and quantum groups (South Hadley, MA, 1992)*, volume 175 of *Contemp. Math.*, pages 193–217. Amer. Math. Soc., Providence, RI, 1994.

[KR08] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. II. *Geom. Topol.*, 12(3):1387–1425, 2008.

[Krc] David Krcatovich. A restriction on the alexander polynomials of L-space knots. arXiv:1408.3886.

[LÔ8] Thang T. Q. Lê. Strong integrality of quantum invariants of 3-manifolds. *Trans. Amer. Math. Soc.*, 360(6):2941–2963, 2008.

[Lew14] Lukas Lewark. Rasmussen's spectral sequences and the $\mathfrak{sl}_N$-concordance invariants. *Adv. Math.*, 260:59–83, 2014.

[LM] Tye Lidman and Ciprian Manolescu. Floer homology and covering spaces. arXiv:1603.00584.

[LOT13] Robert Lipshitz, Peter Ozsváth, and Dylan Thurston. A faithful linear-categorical action of the mapping class group of a surface with boundary. *J. Eur. Math. Soc. (JEMS)*, 15(4):1279–1307, 2013.

[Lur09] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.

[Man14] Ciprian Manolescu. An untwisted cube of resolutions for knot Floer homology. *Quantum Topol.*, 5(2):185–223, 2014.

[MK14] V. D. Mazurov and E. I. Khukhro. Unsolved Problems in Group Theory. The Kourovka Notebook. No. 18 (English version). *ArXiv e-prints*, January 2014. https://arxiv.org/abs/1401.0300v10.

[Ni14] Yi Ni. Homological actions on sutured Floer homology. *Math. Res. Lett.*, 21(5):1177–1197, 2014.

[OS03] Peter Ozsváth and Zoltán Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Adv. Math.*, 173(2):179–261, 2003.

[OS04a] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and knot invariants. *Adv. Math.*, 186(1):58–116, 2004.

[OS04b] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math. (2)*, 159(3):1027–1158, 2004.

[OS04c] Peter Ozsváth and Zoltán Szabó. On Heegaard Floer homology and Seifert fibered surgeries. In *Proceedings of the Casson Fest*, volume 7 of *Geom. Topol. Monogr.*, pages 181–203. Geom. Topol. Publ., Coventry, 2004.

[OS05] Peter Ozsváth and Zoltán Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.

[OS06] Peter Ozsváth and Zoltán Szabó. Lectures on Heegaard Floer homology. In *Floer homology, gauge theory, and low-dimensional topology*, volume 5 of *Clay Math. Proc.*, pages 29–70. Amer. Math. Soc., Providence, RI, 2006.

[OS08] Peter Ozsváth and Zoltán Szabó. Holomorphic disks, link invariants and the multi-variable Alexander polynomial. *Algebr. Geom. Topol.*, 8(2):615–692, 2008.

[PP09] Luisa Paoluzzi and Joan Porti. Hyperbolic isometries versus symmetries of links. *Topology Appl.*, 156(6):1140–1147, 2009.

[Ras03] Jacob Andrew Rasmussen. *Floer homology and knot complements*. ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–Harvard University.

[Ras15] Jacob Rasmussen. Some differentials on Khovanov-Rozansky homology. *Geom. Topol.*, 19(6):3031–3104, 2015.

[Rob01] Rachel Roberts. Taut foliations in punctured surface bundles. II. *Proc. London Math. Soc. (3)*, 83(2):443–471, 2001.

[RR] Jacob Rasmussen and Sarah Dean Rasmussen. Floer simple manifolds and L-space intervals. arXiv:1604.03466.

[Rud83] Lee Rudolph. Algebraic functions and closed braids. *Topology*, 22(2):191–202, 1983.

[Rud01] Lee Rudolph. Hopf plumbing, arborescent Seifert surfaces, baskets, espaliers, and homogeneous braids. *Topology Appl.*, 116(3):255–277, 2001.

[Say98] Nabil Sayari. The reducibility of surgered 3-manifolds and homology 3-spheres. *Topology Appl.*, 87(1):73–78, 1998.

[Spa14] Gilberto Spano. *Knot invariants in embedded contact homology*. PhD thesis, Université de Nantes, 2014.

[SW02] Daniel S. Silver and Susan G. Williams. Mahler measure, links and homology growth. *Topology*, 41(5):979–991, 2002.

[Sza15] Zoltán Szabó. A geometric spectral sequence in Khovanov homology. *J. Topol.*, 8(4):1017–1044, 2015.

[Tau87] C. Taubes. Gauge theory on asymptotically periodic 4-manifolds. *J. Differential Geom.*, 25:363–430, 1987.

[VS99] Luis Gerardo Valdez Sánchez. Dehn fillings of 3-manifolds and non-persistent tori. *Topology Appl.*, 98(1-3):355–370, 1999. II Iberoamerican Conference on Topology and its Applications (Morelia, 1997).

[Web13] Ben Webster. Answer to "khovanov-rozansky homology and spectral sequences" on mathoverflow, 2013. https://mathoverflow.net/a/124925.

[Wit12] Edward Witten. Fivebranes and knots. *Quantum Topol.*, 3(1):1–137, 2012.

[Wu12] Zhongtao Wu. On mapping cones of Seifert fibered surgeries. *J. Topol.*, 5(2):366–376, 2012.